FRACTAL PHENOMENA IN DISORDERED SYSTEMS

R. Orbach

Department of Physics, University of California, Los Angeles, California 90024

INTRODUCTION

Fractal phenomena in disordered systems are now so general that a review of the past year’s developments covers a broad category of topics. These include the formation of particulate aggregates, with implications for electrochemical deposition, interfaces, electrodes, contacts and membranes; dynamics of silica aerogels, and the first direct observation of quantized lattice vibrations of a fractal network (fractons); mechanical simulations of many-body model systems, and the study of nonequilibrium processes like aggregation and fracturing; multifractals, measures with the special property of self-similarity, and their use for the study of random multiplicative processes; earthquakes and fractals and the stochastic behavior of the stress prior to an earthquake; and a newly discovered phenomenon of immense importance to all of us, the fractal mechanisms of cardiac stability.

This compendium of short articles should provide the reader with a good sense of the remarkably diverse and important applications of the notion of fractal geometry. For those who wish more detailed information in a specific area, the references at the end of each article can be consulted.

The articles are grouped according to rather broad areas of interest, with the last two (on earthquakes and cardiac rhythms) focusing on newly developing fields of major personal importance. Thus, articles two through six concern diffusion limited aggregation from a variety of perspectives; seven presents very new X-ray, neutron, and optical scattering evidence

---

for the existence of fractons; eight is an example of how many-body processes in fractals can be studied with mechanical simulations; and nine and ten precisely define multifractals and apply them to random multiplicative systems.

DIFFUSION LIMITED AGGREGATION

[Paul Meakin, E. I. du Pont de Nemours & Company]

The diffusion-limited aggregation model (1) (DLA), in which particles are added one at a time to a growing cluster or aggregate of particles via random walk trajectories, provides a basis for understanding a variety of phenomena of importance in materials science and related areas. These include electrodeposition, fluid-fluid displacement in Hele-Shaw cells (2) and porous materials, dielectric breakdown, random dendritic growth, the dissolution of porous materials, and perhaps other processes such as the growth of cracking patterns and the growth of blood vessels and nerve

![Figure 1](image_url)

Figure 1  A $10^6$ particle off-lattice two-dimensional DLA cluster.
Figure 2  A projection onto a plane (a) and a cross section through the origin (b) for a $3 \times 10^6$ site cubic lattice DLA cluster.
cells. Short reviews of these applications of the DLA model can be found in References (3–5).

During the past year S. Tolman & P. Meakin (6) have developed improved algorithms for both off-lattice and hypercubic lattice diffusion-limited aggregation (DLA) in (Euclidean) dimensionalities 3–8. In addition, an improved two-dimensional off-lattice algorithm was developed. For Euclidean dimensionalities \(d\) of 3–8 the value obtained for the effective fractal dimensionality \(D_a\) was measured using the dependence of the cluster radii of gyration \(R_g\) on the cluster size \(s\) assuming that

\[
R_g \sim s^{(1/D_a)}.
\]

These values were consistent with several mean field theories that predict that the fractal dimensionality \(D\) is given by (7–9)

\[
D = (d^2 + 1)/(d + 1).
\]

On the other hand, for \(d = 2\), a value of 1.71–1.715 was obtained for \(D_a\) using several hundred \(10^6\) particle off-lattice DLA clusters. This result is not consistent with the mean field theory predictions Equation 2. There is no indication that this value changes as we go to larger and larger clusters, and \(D_a\) for two-dimensional off-lattice clusters appears to be by far the best determined fractal dimensionality for DLA. Even if the mean field theory does give the correct results for \(d \geq 3\), a more complete theoretical understanding is still needed.

Most of these results were obtained (1) using quite large numbers (about 100) of clusters containing about \(10^5\) particles or sites (except for \(d > 6\) where smaller clusters were used). In addition, clusters containing \(3 \times 10^6\) sites were grown on cubic lattices (Figure 2) and clusters containing \(10^6\) sites were grown on four-dimensional hypercubic lattices. The overall shapes of these clusters clearly show the effects of the weak lattice anisotropy.

The challenge of developing a complete theoretical understanding of this simple (to define) model has not yet been fully met. This should remain an active area of interest during the next few years.

**Literature Cited**

MULTIFRACTALITY, SCALING LAWS, AND DIFFUSIVE GROWTH

[T. C. Halsey, University of Chicago]

The formation of patterns in physical systems is often dominated by diffusive effects. In 1981, Witten & Sander introduced a simple model, termed diffusion-limited aggregation (DLA), which has, with its variants, successfully described the structures generated in many physical processes, including electrodeposits, viscous fingers, and crystals formed in solidification from a melt (1). In its simplest version, DLA can be described using an electrostatic analogy. Suppose that the surface of the growing object is an equipotential of an electric potential arising from a distant source. The DLA growth law is then equivalent to that obtained by stipulating that the growth probability at a point is proportional to the normal electric field at that point.

An attractive phenomenological approach to DLA is provided by the study of the moments of this normal electric field integrated over the entire surface. If the normal electric field at the point \( s \) is \( P(s) \), then it is found that

\[
\int ds [P(s)]^q = \left( \frac{a}{r} \right)^{\tau(q)}
\]

where \( a \) is the smallest length scale in the problem, and \( r \) is the size of the growing structure \( (2a,b) \). A nontrivial dependence of the exponent function \( \tau(q) \) upon \( q \) reflects the multifractal character of DLA, the fact that the description of its physics requires not one, but an infinite number of fractal dimensions.

The function \( \tau(q) \) can be transformed by a simple operation into a conjugate function \( f(\alpha) \), which possesses an interpretation in terms of local scaling properties of the electric field near the surface of the cluster. Here the exponent \( \alpha \) describes the local scaling of the electric flux on the surface of the cluster, and \( f(\alpha) \) describes the frequency with which that particular scaling occurs. This interpretation makes the latter quantity physically useful in the description, and attempted theoretical derivation, of the physics of diffusive growth.

The electrostatic interpretation of diffusive growth has recently been exploited to obtain scaling laws relating the functions \( \tau(q) \) [or \( f(\alpha) \)] to the fractal dimension \( D \) of the underlying structure (3). Using simple electrostatic identities arising from conservation of energy and some scaling arguments, one arrives at two scaling laws, one of which has been previously derived in a more restricted form by Turkevich & Scher (4), and one of which is new. The first of these relates the dimension \( D \) to the
scaling exponent $\bar{\alpha}$ of the strongest electric field singularity on the surface. In two dimensions, it has the form

$$D = 1 + \bar{\alpha}.$$  

The other scaling law relates $D$ to the third moment of the electric field distribution $\tau(3)$. Its two-dimensional form is

$$D = \tau(3).$$

Both of these relations are in good agreement with available numerical evidence. They can further be extended (in slightly altered form) to higher dimensionalities, as well as to models in which the growth goes as some different power of the local electric field. These latter models have been used to describe the phenomenon of dielectric breakdown.

The problem of the calculation of quantities such as $\tau(q)$ from first principles, despite the constraints provided by the scaling laws above, remains unsolved. Nevertheless, the derivation of scaling laws involving these quantities strongly suggests that they provide at least a foundation for a more systematic approach.

**Literature Cited**


**THE DENSE RADIAL PATTERN IN ELECTROCHEMICAL DEPOSITION**

*[L. M. Sander and David G. Grier, University of Michigan]*

Since the transition from disorderly fractal growth to the formation of ordered dendrites was first observed in electrochemical deposition (ECD) (1a,b), intense study has focused on the structures that form in the range of experimental conditions between those two extremes. These intermediate structures display a large number of highly disordered branches often contained within a sharp envelope. We call these dense radial, dense branching, or simply dense aggregates. Perhaps the most striking property of dense aggregates is the stability of the envelopes that enclose their branches. We have investigated this phenomenon for two-dimensional ECD systems in which the cathode is centered within a ring anode (2) and in which the anode and cathode are separated by a channel. We believe
that the envelopes in such systems are stabilized by the small but nonzero resistance of the growing aggregates.

Previous analyses suggest that an interface should only be stable against long wavelength perturbations if the resistivity of the aggregate exceeds that of the surrounding electrolyte. We measure the resistivity of typical aggregates to be typically less than 20% of the resistivity of the electrolyte, which should be insufficient for stabilization. If, however, we model an aggregate as a disk of uniform resistivity in which currents are constrained to flow in filaments, then we find for the growth rate of mode $m$:

$$x_m = \frac{\delta_m R}{\delta_m v_0} = \frac{m(\beta - 1)}{x^{2m+1} - \beta 1 n \frac{x}{x_0}} - 1$$

where $R$ is the size of the cell, $x$ is the fraction of the cell's radius taken up by the aggregate, $\beta$ is the ratio of the resistivity of the aggregate to that of the solution, and $x_0$ is the ratio of the radii of the cathode and anode. Note in particular that for $\beta > \beta_{\text{min}} \approx [1 + \ln(1/x_0)]$, the interface should be stable. Indeed, no aggregates that have failed this stability criterion have shown any sign of stability (Figure 3).

Figure 3  Zinc aggregate displaying dense radial geometry, 0.03 MZnSO$_4$, 10.0 V.
The Laplace limit is appropriate for much of our work in ECD, because the current through our cells responds linearly to small ripples superimposed on the driving voltage and because the shape of an electrodeposit’s envelope is sensitive to the shape of the anode. Electrodeposition can only be field-driven, or Laplacian, if contributions to the current from concentration gradients and large scale fluid motion are small. We have tracked fluid motion in our system by tracing the path of small (2–20 μm) glass particles suspended in the electrolyte. We observe vigorous convection rolls whose diameter is approximately equal to the thickness of the electrolyte film. This short-scale convection could keep local concentration gradients small without contributing significantly to the overall conduction.

Sensitivity to the anode shape is demonstrated by another as yet unexplained effect within the dense morphology. Over a range of conditions (the exact range depends on the geometry of the ECD cell) copper electrodeposits exhibit a sudden transition at approximately half the dimension of the ECD cell. This transition manifests itself as a change in areal density,

Figure 4 Copper aggregate displaying Hecker Effect: radial geometry.
a change in the metallic luster of the deposit, or both, and is known as the Hecker Effect (Figure 4). In a cell with a triangular ring anode, the outline of the Hecker crossover is a triangle aligned with the anode. The distortion of the overall envelope, on the other hand, is anti-aligned with the anode, as would be expected from field-strength considerations. The Hecker Effect does not arise in zinc, and so far has only been observed in copper.

This work was supported by the National Science Foundation through Grant number DMR8505474.

Literature Cited
1b. Sawada, Y., Dougherty, A., Gollub, J.

SELF-AFFINE FRACTAL SCALING OF THE INTERFACE IN RANDOM BALLISTIC DEPOSITION

[Rémi Jullien, Université de Paris-Sud, France]

In the spirit of the kinetic models for aggregation (1), there exist lattice models of the growing interface in random deposition (2). However, a realistic description of the structure of the deposit can only be done off-lattice. Recently, off-lattice models in two (3) and three dimensions (4) have been introduced, including some restructuring of the deposited particles, and the scaling properties of the interface have been compared with lattice models. In these models, spherical particles (or disks in two dimensions) are added, one at a time, to a growing deposit within a vertical strip of area \( L \times L \) (or width \( L \) in two dimensions), with periodic boundary conditions at the edges of the strip, via random vertical trajectories. The thickness of the surface \( \sigma \) is shown to depend on the width \( l \) and the height \( h \) of the deposit, via the scaling relationship (2), \( \sigma = l^a f(h/l^b) \), where \( f(x) \) tends to a constant when \( x \) tends to infinity and \( f(x) \) scales as \( x^\beta \) with \( \beta = \alpha/\gamma \) when \( x \) tends to zero. This implies a self-affine fractal character for the surface of the deposit.

In two dimensions, three different models have been introduced. In model one, no restructuring is allowed, the falling particle sticks to the deposit at its first contact. In model two, the particle is allowed to rotate about the first contacting particle in the deposit until a second contact is obtained. In model three, the particle is allowed to rotate as often as necessary until it finally reaches a local minimum. The density of
the bulk, estimated from large scale simulation (up to \( L = 16384 \) with 8.107 deposited particles) has been found to be \( \rho = 0.3568 \pm 0.0001, 0.7230 \pm 0.0001, \) and \( 0.8180 \pm 0.0002, \) for model one, two, and three, respectively. The exponent \( \alpha \) is always found to be equal to 1/2, with a quite good accuracy. For the exponent \( \beta, \) the situation is quite clear in model one and two, where a \( \beta \) value very close to 1/3 is recovered as in the lattice ballistic model (5). This result indicates that off-lattice local restructuring does not affect the scaling properties. In the case of model three, it is quite difficult to distinguish between 1/3 and 1/4 for \( \beta. \) The lattice model equivalent to model three gives \( \beta = 1/4 \) (2). This suggests that scaling might be different for lattice and off-lattice versions of the same model.

In three dimensions, four models have been introduced. In model one, no restructuring is allowed. In model two, the particle is allowed to rotate downwards in a vertical plane about the first contacting particle until a second contact is obtained. In model three, the particle is allowed to rotate again about the axis joining the centers of the two contacting particles until a third contact is obtained. In model four, the particle continues to rotate downwards as often as necessary to finally reach a stable position. The density of the bulk (calculated with \( L = 128 \) and \( N = 2^{22} \)) is \( \rho = 0.1465 \pm 0.0003, 0.3790 \pm 0.0003, 0.5231 \pm 0.0005, \) and \( 0.5815 \pm 0.0002 \) for models one, two, three, and four, respectively. The density obtained with model four is appreciably smaller than for compact lattices in three dimensions (0.74048) and is comparable to several experimental values.

The extension to the case of a binary mixture of spherical particles of two different radii, \( R_1 \) and \( R_2, \) has also been investigated (6). The density of the bulk has been studied as a function of the concentration and the polydispersity parameter \( \varepsilon = (R_1 - R_2)/(R_1 + R_2). \) Special attention has been given to the segregation transition that occurs at \( \varepsilon_c = 3^{1/2} - 1 \) when the small particles start to penetrate the random packing of the big ones to large distances. The mean penetration distance of the small spheres \( \Delta Z \) is shown to diverge as \( \varepsilon \) approaches \( \varepsilon_c \) as \( \Delta Z \propto (\varepsilon_c - \varepsilon)^{-\mu}, \) where the exponent \( \mu \) has a value of about 0.55.

At last the study of a lattice model in which there is a finite density of particles in the incoming flux is in progress (R. Jullien, P. Meakin, in preparation).

**Literature Cited**

EXCHANGE ACROSS FRACTAL AND POROUS INTERFACES: ELECTRODES, CONTACTS AND MEMBRANES

[B. Sapoval, Ecole Polytechnique, France]

The question of the response of a rough or porous surface was first considered in the study of the electrical response of electrodes in contact with electrolytes. One uses porous electrodes as a means to increase the surface of the electrodes in high current batteries. In the case of a membrane, neutral reacting species are brought to the surface by diffusion currents instead of electrical current, and the membrane crossing process plays the same role as the redox reaction on the electrode.

The impedance of rough and porous electrodes has long been known to exhibit a high-frequency behavior known as constant-phase-angle (CPA) behavior, i.e. $Z \propto (j\omega)^{-\eta}$ with $0 < \eta > 1$. A recent renewal of interest for this poorly understood problem has followed the proposal by Le Méhauté & Crépy (1) that such electrodes might be described as fractal objects and the CPA exponent $\eta$ be simply related to the fractal dimension $D$ of the electrode. There have been however several conflicting papers about this proposal (1–4).

Detailed calculations of the response of model porous electrodes like the generalized modified Sierpinski electrode have shown that the response depends directly on the electrochemical regime (4). A picture of such an electrode is given in Figure 5. For this parallel structure the frequency response of a blocking electrode is of the CPA type. The exponent depends on the hierarchy but is not a function of the fractal dimension (2, 4). The same exponent describes a nonblocking electrode for which the behavior is governed by pores that have a characteristic size directly related to the frequency. More general models for a porous interface also exhibit CPA behavior (4).

There also exists a diffusion or diffusion and Faradaic regime for which the response depends directly on the fractal dimension. The response is local in contradistinction from the high frequency blocking regime. As a consequence several exponents can be found for different frequency ranges. Starting from low frequency a fractal electrode may present a $\eta = (D-1)/2$ regime (5) followed by a $\eta = D-2$ regime, then followed by a $\eta = 0$ regime, itself followed by a regime where the exponent, indeed, depends...
A generalized modified Sierpinski electrode for which an exact calculation can be made (4): The fractal structure is made by decimation in which at each step, $N$ smaller pores are added. The width and depth of the pores eventually scale with a different factor.

on the hierarchy but not through the fractal dimension. At very high frequencies where edge effects are dominant, a last type of exponent independent of the fractal dimension may appear for the electrode of Figure 5 (4).

In all these cases the admittance of a fractal electrode is not an extensive quantity. It is not proportional to the macroscopic apparent area of the electrode. The response is not proportional to the microscopic transport coefficients either. The admittance presents power-law dependences upon electrolyte resistivity, exchange resistance, specific capacitance, diffusion coefficients, and frequency (4, 6).

The response of rough blocking surfaces has been obtained by considering rough surfaces as perturbed flat surfaces. In first order approximation the relation $\eta = 5 - 2D$ has been predicted (7, 8). It is only in the diffusion regime that the response depends directly on the fractal dimension through the exponent $(D - 1)/2$ whatever the nature, porous, rough, or ramified, of the fractal object.

Because the response is linear, the small signal dc response is directly
related to the small signal ac response at least in the absence of diffusion effects. All the results obtained for the ac response in various geometries in the blocking regime can be connected to the dc response. This permits immediate transposition to the study of constant flux through fractal surfaces, like membranes (6).

Literature Cited


OBSERVATION OF FRACTONS IN SILICA AEROGELS

[R. Vacher, Université des Sciences et Techniques du Languedoc, France]

Silica aerogels are porous structures of SiO₂ glass. They can be prepared with densities as small as 0.02 g/cm³, and the full range from this 99% porosity to bulk silica glass can be covered by changing the preparation conditions. The elements constitutive of the structure are particles, with more or less defined surfaces, whose size a is about 1 nm. Small angle neutron scattering results (1) demonstrate that these particles can assemble to form fractal networks. The fractal geometry extends over more than two orders of magnitude in length scale for the lightest samples. Depending on the pH of the catalyst added to the initial solution, the fractal dimension D can be varied from D = 2.4 under neutral conditions to D = 1.8 under basic catalysis. The correlation length ξ, which defines the upper limit of the fractal structure, ranges approximately from 5 to 100 nm, depending on the density of the gel and on the preparation conditions.

ξ and a define three regimes for the dynamics of these materials. Low frequency phonons, with wavelengths larger than ξ, should propagate as the material is homogeneous at this scale. At larger frequencies, when the wavelength approaches ξ, a crossover to the so-called fractons is predicted by theory (2). On the other hand, a second crossover from fractons to vibrational modes of the particles is expected at high frequencies.

Brillouin light scattering is very well suited for the study of the phonon-
fracton crossover. The position of the peak, and the shape of the line are
determined by the velocity $v$ and lifetime $\tau$ of the acoustic excitation
responsible for the scattering. In the phonon regime, a constant velocity
and $\tau \propto \omega^{-4}$ are expected ($\omega$ is the frequency). For fractons, scaling argu-
ments lead to $v \propto \omega^\alpha$, where $\alpha \approx 0.5$, and $\tau \propto \omega^{-1}$. The experimental spec-
tra give evidence of a strong broadening of the peak as the wavevector is
increased (3). The shape of the spectra was compared with theoretical predictions (4). An analytical expression describing a smooth crossover
from phonons to fractons, in agreement with the limiting regimes, gives
excellent fits of the spectra. From the values obtained for the crossover
frequency, it was demonstrated that Brillouin scattering covers the full
crossover region from propagating phonons to localized fractons. The
dispersion curve for fractons was also determined over more than one
decade in frequency.

The full fracton range was investigated by depolarized Raman scattering
(5). The experiments, performed at unusually low frequencies, cover the
range from 0.3 to 30 cm$^{-1}$. The spectra give evidence of a power-law region, limited on the low-frequency side by the phonon-fracton crossover,
while at high frequency the peak related to the first Raman active mode
of the particles is evident. A value of the spectral dimension of fractons
can be deduced from this power-law dependence, in agreement with the
independent Brillouin scattering determination.

Incoherent, inelastic neutron scattering was used on the same samples
to determine quantitatively the density of vibrational states $Z(\omega)$ (6). The
frequency range from 0.07 to 5 THz was explored in a time of flight
experiment. In the high-frequency region, the observed $Z(\omega) \propto \omega^{1.5}$ agrees
very well with the density of states predicted for small particles. The
decrease of the slope at lower frequencies gives an indication of a crossover
to a fracton density of states. From this experimental $Z(\omega)$, the thermal
properties were calculated, in good agreement with the experiments.

Literature Cited

   Lett. 6: 245
4a. Aharony, A., Alexander, S., Entin-
   Mag. B 56: 949
   Phys. Rev. B 37: 7726
5. Tsujimi, Y., Courtens, E., Pelous, J.,
6. Vacher, R., Woignier, T., Pelous, J.,
   Coddens, G., Courtens, E. 1989. Sub-
   invited for publication
FRACTAL PHENOMENA

FRACTAL STRUCTURES IN MICROSphere SYSTEMS

[Arne T. Skjeltorp, Institute for Energy Technology, Norway]

Uniformly sized polymer microspheres (1) have proved to be very useful as many-body systems to study nonequilibrium processes like aggregation (2-4) and fracturing (5, 6), which lead to disordered and often fractal structures. Direct comparisons of such model studies and computer simulations have also provided a basis for better understanding of processes far from equilibrium (7).

Figure 6 thus summarizes a wide range of experimental (2) and simulated (8, 9) growth patterns for diffusing particles in a plane. In the experiments, water-dispersed polystyrene spheres were confined between closely spaced glass-plates with separation slightly larger than the sphere diameter. The attractive potential energy between the particles could be varied relative to the thermal energy. This was achieved by balancing the short-range attractive van der Waals forces against the net repulsive electrostatic forces between the charged particles controlled by the counter-ion concentration in the dispersion. Figure 6a represents a relatively fast diffusion-limited

![Figure 6](image)

*Figure 6  Top row: Microscope pictures of aggregated spheres [4.7 μm in (a), 1.1 μm in (c)–(d)] for successively slower growth velocity \( v_g \) (defined as the average growth of radius of gyration): (a) ramified clusters \( v_g \approx 10^{-3} \) μm/sec; (b) rough single crystal with holes \( v_g = 3 \times 10^{-4} \) μm/sec; (c) dendritic crystal \( v_g \approx 10^{-3} \) μm/sec; (d) faceted hexagonal crystal \( v_g \approx 5 \times 10^{-6} \) μm/sec. Bottom row: (a) computer simulations for cluster-cluster aggregation including rotations; (b)–(d) DLA growth with increasing effective surface tension.*
cluster aggregation (DLCA) process. The fractal dimension $D$ for this system was found to be $D = 1.49 \pm 0.05$. This is in good agreement with the simulated DLCA result (8), which also allows rotation. Figure 6a produces a fractal dimension $D = 1.685 \pm 0.015$. Figures 6b–d show examples of successively slower particle cluster growth by reducing the attractive interactions between the spheres. This behavior clearly demonstrates that if added particles are given more and more time to find a favorable place on the perimeter of the growing aggregate (reduced sticking probability, migration) the structures become more and more regular. This trend is also nicely reproduced in computer simulations (9) shown in Figures 6b–d.

Similar experiments using diffusing particles interacting via long-range magnetic dipolar forces have shown quite different growth patterns in accordance with computer simulations (4). For this study, $d = 3.6 \mu m$ sulfonated polystyrene spheres containing iron oxide were used. The spheres could be magnetized to various levels of remnant magnetization $M_r$. Figure 7 shows a series of aggregated structures formed after a few hours for increasing $K_{dd} = \mu^2/d^3k_BT$ representing the dipole-dipole interaction relative to the thermal energy [$\mu = M_r(\pi d^3)/6$ = magnetic moment carried by the spheres]. There is an increasing tendency for the spheres to form chains and open loops as $K_{dd}$ increases. This is also reflected in the decrease in the fractal dimension $D$, Figure 8.

Quite different uses of microspheres have involved model studies of fracture (5) in direct comparison with computer simulations (6). For this study, mono-sized sulfonated polystyrene spheres (3.4 $\mu m$) dispersed in water were packed to a dense monolayer between planar glass plates. It

![Figure 7](image)

*Figure 7* Aggregates of 3.6 $\mu m$ spheres for increasingly reduced magnetization: (a) $K_{dd} = 16$; (b) $K_{dd} = 100$; (c) $K_{dd} = 1360$. 
was thus possible to form a polycrystal with relatively large crystalline grains containing typically $10^5$–$10^6$ spheres. The fracturing was realized as a result of particle shrinking during a slow drying process reducing the sphere diameter to 2.7 $\mu$m. This produced breaking of the bonds between the spheres but not of the weaker bonds between the spheres and the glass surface (no buckling).

**Figure 8** (a) Fractal dimension $D$ versus dipolar coupling constant $K_{dd}$. $D = 1.49 \pm 0.06$ is the DLCA value for nonmagnetic spheres. For large $K_{dd}$, $D$ approaches the simulated value $D = 1.23 \pm 0.12$. The solid line is only a guide to the eye.

**Figure 9** Final fracture pattern in a monolayer of spheres for which the diameter has been reduced from 3.4 $\mu$m to 2.7 $\mu$m.
The evolution of fracture in the whole sample showed many characteristic features. The first cracks were formed along the grain boundaries where fewer bonds had to be broken than inside the grains. The cracks inside the grains propagated rapidly more or less linearly. As more and more cracks were formed, they became more and more irregular, reflecting the irregular strain field in the sample. A typical final crack pattern inside one grain is shown in Figure 9. The visual appearance of this suggests that the cracking patterns might be described in terms of the concepts of fractal geometry. In fact, digital analysis of the cracks for a wide range of length scales produced an effective fractal dimensionality $D = 1.68 \pm 0.05^5$.

**Literature Cited**


**MULTIFRACTALS**

[B. B. Mandelbrot, I.B.M., Thomas J. Watson Research Center]

Multifractals have lately become an important tool in physics. Recent work by B. B. Mandelbrot streamlines and amplifies his pioneer work of 1968–1976. It brings a very promising extension of their range of practical applicability, as well as clear improvements to their theory. It is important to begin by stressing that multifractals are not sets; they are measures having special properties, e.g. self-similarity. One way to help intuition is to identify "measure" with "probability measure": When a point is chosen at random on a set, the measure of a subset (a portion of that set) is the probability of the point falling in this subset. Another way to help intuition is to identify “measure” with “limit frequency.” When the position of a dynamic system visits a set repeatedly, the measure of a subset is the limit of the frequency with which the visit occurs within that subset. To take an example, a measure on the Sierpinski gasket is strictly self-similar if the
following is the case: the 3 thirds of the overall gasket are assigned pre-
scribed measures \( m_0, m_1 \) and \( m_2 \), adding to 1. Then the measure in each
third is subdivided among its 3 ninths in the same relative proportions \( m_0, m_1 \)
and \( m_2 \). And so on recursively. The resulting multifractal measure is
called multinomial by Mandelbrot.

The simplest of all multifractal measures is supported by the interval
\([0, 1]\) and is called binomial. To construct it, given \( m_0 \) satisfying
\( \frac{1}{2} < m_0 < 1 \) and \( m_1 = 1 - m_0 \), we spread mass over the halves of every
dyadic interval, with the relative proportions \( m_0 \) and \( m_1 \). If \( t = 0, \eta_1, \eta_2, \ldots, \eta_k \)
is the development of \( t \) in the binary base 2, and \( \varphi_0 \) and \( \varphi_1 \) the relative
frequencies of 0s and 1s in the binary development of \( t \), the binomial
measure assigns to the dyadic interval \([dt] = [t, t + 2^{-k}]\) of length \( dt = 2^{-k} \)
the mass \( \mu(dt) = m_0^{\varphi_0} m_1^{\varphi_1} \). The Hölder exponent is then

\[ \alpha = \log \frac{\mu(dt)}{\log (dt)} = -\varphi_0 \log_2 m_0 - \varphi_1 \log_2 m_1. \]

The number of intervals leading to \( \varphi_0 \) and \( \varphi_1 \) is \( (\kappa \varphi_0)! (\kappa \varphi_1)! / \kappa! \), giving
the box fractal dimension \( \delta = \log \frac{(\kappa \varphi_0)! (\kappa \varphi_1)! / \kappa!}{\log (dt)} \). For large \( \kappa \),
the Stirling approximation yields \( \delta = -\varphi_0 \log_2 \varphi_0 - \varphi_1 \log_2 \varphi_1 \). Thus, \( \alpha \)
determines \( \varphi_0 \), hence \( \delta = f(\alpha) \).

To construct a multinomial measure of base \( b > 2 \), we require \( b \) masses
\( m_\beta \) (\( 0 \leq \beta \leq b - 1 \)). The \( b \)-adic intervals characterized by the frequencies
\( \varphi_\beta \) of the digits \( \beta \) in the base-\( b \) development \( 0, \eta_1, \eta_2, \ldots, \eta_k \) yield

\[ \alpha = -\sum \varphi_\beta \log_b m_\beta \quad \text{and} \quad \delta = -\sum \varphi_\beta \log_b \varphi_\beta. \]

The collections of \( b \) numbers \( \varphi_\beta \)'s yielding the same \( \alpha \) are dominated by
the highest dimension term. This term maximizes \( -\Sigma \varphi_\beta \log_b \varphi_\beta \), given
\( -\Sigma \varphi_\beta \log_b m_\beta = \alpha \), and \( \Sigma \varphi_\beta = 1 \). The classical method of Lagrange
multipliers introduces a multiplier \( q \), with \( -\infty < q < \infty \), and yields

\[ \varphi_\beta = \frac{b^q \log_b \sum m_\beta}{\sum b^q \log_b \frac{m_\beta}{m_\beta}} = \frac{m_\beta^q}{\sum m_\beta^q}. \]

In terms of \( \tau(q) = -\log_b \sum m_\beta^q \), which the mathematical statisticians call
cumulant generating function, the Lagrange multipliers determine \( q \) and
\( f(\alpha) \) from \( \alpha \) by

\[ \alpha = -\sum \varphi_\beta \log_b m_\beta = -\frac{\partial}{\partial q} \log_b \sum m_\beta^q; \]

\[ \max \delta = f(\alpha) = -\frac{\sum (q \log_b m_\beta - \log_b \sum m_\beta^q) m_\beta^q}{\sum m_\beta^q}. \]
That is,
\[ \alpha = \frac{\partial \tau(q)}{\partial q} \quad \text{and} \quad f(\alpha) = q \frac{\partial \tau}{\partial q} - \tau = q\alpha - \tau. \]

Formally, \( q \) = inverse temperature, \( \tau \) = Gibbs free energy, and \( f \) = entropy. These interpretations do not come in later after the fact, but are implicit in the derivations.

When B. B. Mandelbrot first investigated the multifractals, from 1968 to 1976, the most readily available work being his paper (1), he stressed the effects of a major empirical constraint of observation. The multifractals one really wishes to theorize upon are supported by real three-dimensional physical space (e.g. turbulent dissipation), or sit on a strange attractor in the high-dimensional phase space of a dynamical system. On the other hand, often multifractals can only be observed via cuts: for example, via one-dimensional cuts through a field of turbulent dissipation, or the Poincaré sections in dynamics. These cuts or sections are for all practical purposes random, even if the overall multifractal is not. To tackle them, Mandelbrot has developed a generalization of the multinomial multifractal, called 1974 multifractals, to which the same formation applies (2). The most important tool in their study is Cramér's theory of large deviation limit theorems, and the most striking feature is that the function \( f(\alpha) \) can now be negative.

**Literature Cited**


**RANDOM MULTIPLICATIVE PROCESSES AND MULTIFRACTALS**

[S. Redner, Boston University]

**Introduction**

Random multiplicative processes abound in a variety of natural phenomena (1, 2). However, their essential properties are not as well appreciated as random additive processes (such as a random walk). For this case, the central limit theorem provides crucial information about the asymptotic properties of a sum of random variables. Namely, short-range correlations in the sequence of variables do not affect the asymptotics of the sum, and
one-parameter scaling holds. On the other hand, multiplicativity gives rise to multifractal scaling, as seen in a variety of contexts (3–7).

I first discuss a simple random multiplicative process, in order to introduce new features not present in random additive processes, and then use these results to relate the scaling properties of (apparently disjoint) realizations of random multiplicative processes. For the distribution of voltages in a random resistor network, a good qualitative picture is provided by an uncorrelated multiplicative process, which leads to a log-binomial voltage distribution and to multifractal scaling. A random walk in a medium containing traps and sources provides an understanding of the role of short-range correlations in a multiplicative process, which stems from the recurrence (or transience) in the underlying random walk. For the kinetics of fragmentation, the general conditions under which the fragment size distribution is described by the log-normal form, and by a conventional scaling distribution, are examined.

A Pedagogical Example

For a sequence of \( N \) independent, identically distributed random variables consisting of the numbers 2 and 1/2 with equal probability, the average value of the product \( \langle P \rangle = (5/4)^N \). However, the most probable value, \( P_{mp} \), which arises when equal numbers of 2s and 1/2s appear in the sequence, equals 1. Note also that if 1/2s are twice as likely to occur as 2s in the sequence, then \( \langle P \rangle = 1 \), while \( P_{mp} = [(1/2) \cdot (1/2) \cdot 2]^{N/3} = 2^{-N/3} \). The higher moments of the product, \( \langle P^\kappa \rangle \), scale as \( \langle P^\kappa \rangle = (2^{\kappa-1} + 2^{-\kappa-1})^N \). That is, \( \langle P^\kappa \rangle \) cannot be written as \( \langle P^\kappa \rangle \sim \exp [a(\kappa)N] \), with \( a(\kappa) \) strictly linear in \( \kappa \). This loss of scaling stems from the long tail in the underlying distribution.

The essential reason for the disparity between \( \langle P \rangle \) and \( P_{mp} \) is rare events. A sequence consisting of all 2s occurs with an exponentially small probability, but the value of the product is exponentially large. This extreme event thus makes a finite contribution to \( \langle P \rangle \), and dominates in the higher moments of the product. This dominance by rare events means that numerical simulations of a random multiplicative system will detect the most probable value unless one has the resources to observe a finite fraction of all the states.

It is convenient to study in a random multiplicative process the continuum limit, and this leads to the classical log-normal distribution. This is a poor approximation, however, because the average value and higher moments of the product are dominated by the events at the tail of the distribution, and these are inadequately described by the Gaussian approximation. Thus one must be aware of the limits of this approximation, when applied to multiplicative processes.
An additional interesting feature is the sensitivity of $\langle P \rangle$ to short-range correlations in the sequence of variables. Suppose, e.g. that there are no immediate reversals in the sequence. That is, when a 2 first appears, the next element must also be a 2. Only after the second appearance of a 2 does the sequence become uncorrelated again. This nearest-neighbor correlation is equivalent to replacing the correlated sequence of 2s and 1/2s, by an uncorrelated sequence of $N/2$ variables, which may be either 4 or 1/4. Thus we find $\langle P \rangle_{\text{corr}} = (\sqrt{17/8})^N \gg (5/4)^N$ as $N \to \infty$. The increase in $\langle P \rangle_{\text{corr}}$ compared to the uncorrelated process stems from the relatively larger role played by rare events. This shows that there is no analog of a central limit theorem for a multiplicative process.

Voltage Distribution in Random Resistor Networks

The conductance of a random resistor network can be expressed the second moment of the distribution of voltage drops across the bonds in the system, $v_{ij}$, as $G = \Sigma v_{ij}^2$, where the conductance of each bond has been taken to be unity. However, fundamental information is contained in the distribution of voltage drops itself (5, 6, 8). This distribution is connected with the geometrical structure of the conducting backbone. Near the percolation threshold, this backbone is self-similar, containing both blobs and singly connected bonds on all length scales. A simple hierarchical model, in which a single bond is iteratively replaced by a unit cell consisting of 2 bonds in parallel and 2 bonds in series (5) captures these essential features. From the voltage distribution on this model, one sees that the individual bond voltages change in a multiplicative fashion upon length rescaling. The voltage distribution in an $n$th-order hierarchy equals the coefficient of $V = V_1 V_2^{-j}$ in the binomial expansion of $[2(V_1 + V_2)]^N$, where $V_1 = 1/5$, and $V_2 = 2/5$ for the hierarchical model. Thus the voltage distribution is a log-binomial, and this gives rise to multifractal scaling properties, in agreement with observations on random resistor networks at the percolation threshold.

Random Walk in a Random Multiplicative Environment

Consider a random walk in a random medium containing partially absorbing traps of strength $t$ and also sources of strength $s$. The total number of walkers after $N$ steps is governed by random multiplication. However, the sequence of $ts$ and $ss$ in a random walk trajectory will be correlated, reflecting the propensity for the walker to visit a particular defect many times before reaching another defect. These correlations in the first passage probability determine the distribution of particles after $N$ steps.

For a periodic distribution in which a single source is immediately followed by 2 traps on a one-dimensional chain, all possible outcomes
for $N$-step walks are generated by the matrix product $M^N(i_0)$, where $M = (\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix})$. The corresponding generating function is $G(s, t) = (1) (1-M)^{-1}(i_0) = (1+t)/(1-t-2st)$. The $N$th term in the series representation of $G(s, t)$ yields the number of walkers at the $N$th step. For the simple case $s = 2$ and $t = 1/2$ (discussed above), the average value of the product is unity if the factors in the product are uncorrelated. For the random walk, however, the average number at the $N$th step, i.e., the average of the product of $ts$ and $ss$, varies as $\mu^N$, with $\mu = (\sqrt{33}/16 - 1/4)^{-1} \approx 8431$. The difference between the two results stems from the relatively higher probability of generating a sequence consisting of all $ts$ in the random walk than in the uncorrelated process.

**Fragmentation Processes**

Fragmentation is a ubiquitous kinetic phenomenon that underlies processes such as the crushing of rocks. For this case, the distribution of fragment sizes can have a log-normal form that arises from the multiplicative size reduction of a fragment in a single breakup event. We study the general conditions on the nature of a breakup event that determines whether the distribution of fragment sizes is log-normal, or whether it assumes a scaling form.

The rate equations for fragmentation are (9)

$$\frac{\partial}{\partial t} c(x, t) = -a(x)c(x, t) + \int_x^\infty c(y, t)a(y)f(x, y)dy,$$

where $c(x, t)$ is the concentration of $x$-mers at time $t$, $a(x)$ is their overall rate of breakup, and $f(x|y)$ is the rate at which $x$ is produced from the breakup of $y$. We consider homogeneous kernels for which $a(x) = x^\alpha$. This also implies that $f(x|y) \sim y^{-\beta}(x/y)$, while mass conservation imposes $\int_0^1 x b(x) dx = 1$.

For the distribution of fragment sizes in the small size limit, there are two generic cases (10). One is where the reduced breakup kernel, $b(x)$, has a sharp cutoff at a non-zero value of $x$, corresponding to a lower limit on the relative reduction of a fragment in a single breakup event. The second case is that of no cutoff, e.g., $b(x) \sim x^\alpha$, as $x \to 0$. To find the small size behavior, we make the scaling ansatz for the cluster size distribution, $c(x, t) \sim s^{-2}\phi(x/s)$, where $s$ is a typical cluster size (11, 12). Substituting these distributions into the rate equations, and then computing moments of the resulting scaling equation leads to $m_{z+\alpha} = \omega(1 - \alpha/L_z - 1)m_z$, where $m_z = \int_0^\infty x^\alpha\phi(x)dx$, and $L_z = \int_0^1 x^\alpha b(x) dx$, are the moments of the scaling function and the reduced breakup kernel, respectively, and $\omega$ is a constant. A detailed calculation shows that for a sharp cutoff, a log-normal tail in
the cluster size distribution results, while if there is no cutoff, a power-law distribution is obtained.

Conclusions

A variety of statistical mechanical problems are governed by random multiplicative processes. The theory of such processes is much less developed, and also less well-appreciated than the theory of random additive processes, such as random walks. Multiplicativity appears to be an essential ingredient in leading to multifractal scaling properties. The study of correlated multiplicative processes and their corresponding scaling properties should prove an interesting new area for further investigations.

Literature Cited


EARTHQUAKES AND FRACTALS

[Y. Y. Kagan, University of California, Los Angeles]

Several catalogs of tectonic earthquakes have been analyzed statistically to study interrelations between earthquakes for further theoretical understanding of the earthquake process. As first approximation, an earthquake may be represented by a sudden shear failure—appearance of a large dislocation loop (1). The catalogs characterize the set of earthquakes by origin time, hypocenter position, and by the second-rank seismic moment tensor for each event. Earthquake sequences have been treated as a tensor-valued multidimensional stochastic point process. Statistical properties of an earthquake occurrence exhibit scale-invariant features: the Pareto distribution of the earthquake size (magnitude or scalar seismic moment), power-law decay of the rate of the aftershock and foreshock occurrence (Omori’s law), self-similarity of the earthquake occurrence rate in different magnitude ranges (2, and references therein). The two-, three-, and four-point moment functions of the spatial distribution of shallow earthquake
hypocenters have also been studied. The results indicate that earthquake hypocenters are concentrated on a set with the fractal dimension between 2.0 and 2.25 (3, 4). The results appear to rule out the conventional model of earthquake hypocenters occurring on a single isolated plane or on several planes, and require, instead, that an earthquake fault zone be nonplanar and fractal.

The observed self-similar patterns of an earthquake occurrence are shown to be derived from simple assumptions of the stochastic behavior of the stress prior to an earthquake. In time domain, Omori's law of foreshock/aftershock occurrence and, in general, the time clustering of earthquake events have been shown to be a consequence of branching of earthquake fractures and Brownian motion-like behavior of random stresses (5). This means that the fractal dimension of temporal occurrence of earthquake sequences is equal to 0.5. Elaborating on Zolotarev's result (6, section 1.1), we derive spatial patterns of earthquake fault branching from the assumption of the presence of random point defects in rocks and other materials (7). These defects rule out the planar propagation of a fault rupture. The deviations of the fault surface from planarity are characterized by a rotational Cauchy distribution that, as we have found previously, explains the nonplanar geometry of fault systems for natural earthquakes. We have demonstrated that as a result of the 3-D rotations of elementary dislocations comprising the earthquake source zone, the resulting complex extended source should contain rotational dislocations (disclinations) that we identify with asperities/barriers controlling the initiation, propagation, and stopping of earthquake fractures (8).

On the basis of stochastic modeling, a model of an earthquake occurrence has been developed to simulate the earthquake process by the Monte-Carlo method (2, 8, and references therein). In this model an earthquake fault pattern is assumed to consist of a system of small elementary plane dislocations. Following the initial dislocation, subsequent ruptures occur according to a stochastic critical branching process. The position of each secondary dislocation is randomly shifted, along the fault-plane of its predecessor, from the location of the main shock. In addition, the orientations of the fault-plane and slip vector of the secondary dislocations are rotated according to the three-dimensional Cauchy distribution. The simulated fault pattern is both visually and statistically similar to real earthquake faults. We use the above results to identify a foreshock sequence in progress in order to develop a method of online earthquake prediction (2). As a predictor, the procedure reduces the average uncertainty in the rate of occurrence for a future strong earthquake by a factor of more than 1000 when compared with the completely random (Poisson) rate of occurrence.
DYNAMICS OF SUDDEN DEATH

[A. L. Goldberger and D. R. Rigney, Harvard Medical School]

Each year, tens of thousands of Americans die suddenly from a disturbance of their heart rhythm (cardiac arrhythmia). New understanding of the mechanisms of both normal cardiac function and sudden cardiac death comes from the application of nonlinear dynamics and fractals to physiology and medicine (1–5). These nonlinear concepts challenge certain deeply ingrained preconceptions regarding the dynamics of healthy function and disease.

Under healthy conditions, the heartbeat (pulse rate) is controlled by the firing of pacemaker cells in the sinus node located in the right atrium. Physicians and physicists often assume that the healthy heart rate is highly regular, as implied by the clinical term regular sinus rhythm. Closer inspection of heart rate time series data from healthy subjects at rest, however, reveals an unanticipated finding, namely, cardiac chaos. The heart rate in normal individuals even at rest fluctuates in an erratic fashion. Further, the spectrum of this process is broadband, with a 1/f-like distribution (Figure 10). Based on these findings we have proposed that heart rate is under the control of a nonlinear, fractal feedback system that generates self-similar fluctuations across multiple orders of temporal magnitude. The details of this fractal mechanism are currently being investigated. An essential feature appears to be the nonlinear interactions between the sympathetic and parasympathetic branches of the autonomic (involuntary) nervous system that regulates the firing rate of sinus node pacemaker cells.

This counterintuitive chaos theory of healthy variability is supported by two additional lines of evidence. The fractal nature of healthy function is
not limited to the 1/f-like spectrum of the normal heartbeat. The fractal concept also applies to certain spatial as well as temporal aspects of physiology. The geometric patterns seen in a variety of complex physiological structures show self-similarity and suggest a fractal principle of morphogenesis. For example, in the heart itself, self-similar branching structures are apparent in the coronary arteries, the chordae tendineae (supporting the mitral and tricuspid valves), the electrical conduction (His-Purkinje) system of the ventricles (Figure 11), and the branching of certain cardiac muscle bundles (3).

A second line of evidence for the fractal theory of cardiac health derives from analysis of dynamic patterns in heart disease. Until recently, it was widely assumed that sudden death and the events preceding cardiac arrest represented a route to chaos. However, work from our laboratory and others suggests that this notion may not be correct (4). When we examined the heart rate dynamics prior to the onset of a fatal arrhythmia, we observed a loss of the physiologic heart rate variability (5). Heart rate usually becomes more—not less—regular before sudden death (Figure 10), associated with a change from a broad to a relatively narrow frequency spectrum. Furthermore, the electrophysiological dynamics actually observed during
Fractal geometry is widespread in anatomic structures. The special conduction system (His-Purkinje network) that transmits electrical impulses from the atria to the ventricular muscle (myocardium) is a self-similar branching network. This fractal-like system is the anatomic substrate for electrical stability of the heart. Adapted from Reference 1.

Cardiac arrest due to ventricular fibrillation and other lethal arrhythmias are also relatively periodic, not chaotic (5). Based on these kinds of findings, we have proposed that "cardiac arrest follows a progression, often involving bifurcations, from the fractal chaotic dynamics of the normal heartbeat to the pathologic periodicities and loss of variability of the dying heart" (4). These findings may have important implications for monitoring patients at high risk of sudden death, as well as for modeling the nonlinear mechanisms of healthy function and the dynamics of diseases of the heart and other organ systems (3).

Literature Cited


CONTENTS

PREFATORY CHAPTER

Ceramic Materials Science in Society, W. David Kingery 1

EXPERIMENTAL AND THEORETICAL METHODS

Dynamical Diffraction Imaging (Topography) with X-Ray Synchrotron Radiation, M. Kuriyama, B. W. Steiner, and R. C. Dobbyn 183

Use of Laser Techniques to Study the Dynamics of Molecular-Surface Interaction, J. Häger and H. Walther 265

Towards Unified Computer Models for Predicting Fracture of Solids, Alan K. Miller 439

Fractal Phenomena in Disordered Systems, R. Orbach 497

PREPARATION, PROCESSING, AND STRUCTURAL CHANGES

Gas-Source Molecular Beam Epitaxy, Morton B. Panish and Henryk Temkin 209


Ion Beam Processing for Surface Modification, J. K. Hirvonen 401


PROPERTIES AND PHENOMENA

Chemically Induced Interface Migration in Solids, Duk N. Yoon 43

Materials Synthesis by Mechanical Alloying, C. C. Koch 121

The Structures of Electrodeposits and the Properties that Depend on Them, Rolf Weil 165

Polymer Interdiffusion, H. H. Kausch and M. Tirrell 341

Structural Relaxation in Metastable Strained-Layer Semiconductors, Brian W. Dodson and Jeffrey Y. Tsao 419
SPECIAL MATERIALS

Fluoride Glasses, J. M. Parker 21
Very Low Thermal Expansion Coefficient Materials, Rustum Roy,
D. K. Agrawal, and H. A. McKinstry 59
Electroluminescent Display Materials, H. Ohnishi 83
Composite Electrolytes, Nancy J. Dudney 103
Hardmetals and Cermets, P. Ettmayer 145
Intermetallic Compounds for Strong High-Temperature
Materials: Status and Potential, R. L. Fleischer, D. M. Dimiduk,
and H. A. Lipsitt 231

Structural Properties of Ionomers, C. W. Lantman,
W. J. MacKnight, and R. D. Lundberg 295
Crystal Chemistry and Properties of Mixed Valence Copper
Oxides, B. Raveau and C. Michel 319
Synthesis, Stabilization, and Electronic Structure of Quantum
Semiconductor Nanoclusters, Michael L. Steigerwald and
Louis E. Brus 471

INDEXES

Subject Index 551
Cumulative Index of Contributing Authors, Volumes 15–19 557
Cumulative Index of Chapter Titles, Volumes 15–19 559