A PROBABILISTIC DESCRIPTION OF THE SEISMIC REGIME*

The aggregate of the natural seismic phenomena is considered as a point random process. The theory of point processes is set forth and the possibilities of extrapolation of the process are examined. Different types of processes which can serve as approximations to the real seismic regime, are investigated; the model of a branching process is described in a more detailed way.

The possibilities of a statistical study of the different parameters of the process are discussed.

INTRODUCTION

It is well known that the seismic regime, i.e., the aggregate of the natural seismic phenomena (earthquake, mountain-slides and seismoacoustic impulses in mines, microdestruction impulses in specimens, etc.) has quite an irregular, random character. The reason for this is the diversity of the conditions and factors influencing the disintegration of rocks and other materials and the difficulties in determining these. Hence, it is natural to apply statistical methods to the aggregate of seismic phenomena in order to bring out certain regular features of the regime from the background of pure random fluctuations. A serious difficulty which comes in the way of a statistical description of the regime, is the fact that the seismic process is essentially multidimensional and hence unidimensional representations of such a process can necessarily bring out only certain of its integral and coarse characteristics. This is an obstacle to an effective application of the known methods of the theory of unidimensional random processes to this field. Hence, in recent times, researches on the seismic regime were, to a large degree, confined only to a study of the partial statistical characteristics of the process: of the energy distribution (frequency law), space distributions (seismic-activity maps), distributions of the time intervals between earthquakes, etc. A detailed and thorough survey of the various research works done in this field is given in [1].

Investigations on the combined space-time and energy characteristics were beset with considerable difficulties due to a lack of a relevant formal probabilistic language for the description of these relationships and a statistical model for the regime. The aim of this paper is to obviate this deficiency in some measure.

From the probability standpoint, the seismic regime can best be treated as a flux of random events in a multidimensional phaseal space or as a random point process. The possibility of such a representation was indicated by B.V. Gnedenko in his introduction to the book of Khintchin [2]. A similar view has been systematically developed by D. Vere-Jones in his article [3], where he has examined the theory of point processes, particularly that which can be applied to the models of event-clustering. In this paper he gives special emphasis to the theoretical and the statistical aspects of the processes which are unidimensional in time. Even though part of the theoretical formulations holds for multidimensional cases as well, the essentially unidimensional treatment of the work renders the practical application of the developed apparatus to a statistical analysis of the seismic regime difficult.

1. RANDOM POINT PROCESSES

In this section we examine the theory of multidimensional random point processes (fields), applicable to the problems concerning the stochastic description of the seismic regime. For an account of this theory we lean heavily upon [4-5].

We shall first examine some random event which can be fully characterized by a point in the multidimensional phaseal space X. Henceforth, we shall mainly confine ourselves, as in [7], to the five-dimensional space $X = H \times T \times E$, where $H$ is the Euclidean three-dimensional space, $T$ is time and $E$ is the energy. A point in such a space will denote the focal coordinates, time and energy of the seismic phenomenon. Spaces of larger or smaller dimensions can be considered in an analogous way. We shall consider the number of phenomena in any limited interval of space $X$ (when $E > 0$) with a probability 1, as finite.

Let us denote by $N(A)$ the number of phenomena in the domain $A$. $N(A)$ is a random integer. $P\{N(A) = n_1, \ldots, N(A_k) = n_k\}$ is the probability that, in the domain $A_1$, there occur $n_1$ events, in $A_2$ there occur $n_2$ events, etc. The following probability is of special importance to us:

$$\frac{1}{n!} P\{N(A_1) = 0, N(dx_1) = 1, \ldots, N(dx_k) = 1\} = p_n^T(dx^*);$$

$$A = A_1 + \sum_{i=1}^{k} d_x_i$$

In this formula $dx_i$ denotes an infinitesimally small interval around the point $x_i$. Thus $p_0^T(dx^*)$ is the probability that, in the domain $A$, exactly $n$ events have occurred, the $i$-th one corresponding to the interval $dx_i$. The composite event $(N(A_1) = 0, N(dx_1) = 1, \ldots, N(dx_k) = 1)$ can be realized in $n!$ ways and hence the coefficient $1/n!$ in formula (1).

We shall now introduce the functions $N(A_1 \times \ldots \times A_k)$, which are equal to the number of selections, without reversing, of $l_1$ events from the domain $A_1$, $l_2$ events from the domain $A_2$, etc., $k = \sum_i l_i$. Then, for example, $N(A_1 \times A_2) = N(A_1) \cdot N(A_2)$, $N(A_1 \cap A_2)$, where $A_1 \cap A_2$ is the intersection of the domains $A_1$ and $A_2$. We shall define the $k$-th order factorial moment as

$$m_n(A_1 \times \ldots \times A_k) = N(A_1 \times \ldots \times A_k),$$

where the line on the top denotes averaging (mathematical expectation).

The use of factorial moments is more suitable than ordinary moments for a description of the point process. The ordinary moments can be easily calculated, if we know the factorials of the same or lower orders (see, for example [6]). Formula (2) shows that $m_k(dx^*) = m_k(dx_1 \times \ldots \times dx_k)$ can be taken as the probability of the phenomena occurring in the intervals $dx_i$, irrespective of the total number and location of the other events, i.e., unlike the functions $p_d(0)$, the factorial moments characterize the local stochastic properties of the process [5]. These functions are connected to one another by the following relationships [4]:

$$p_n^T(dx^*) = \sum_{l=0}^{n} \frac{(-1)^{n-l}}{n! (l-n)!} m_l(dx^* \times R^{l-n}).$$
The primary role in the theory of point random processes is played by the concept of a generating function, which, as is shown in [4], completely defines the entire probability structure of the process. For the bounded function \( \eta(x) \), which is identically equal to unity outside some closed volume of the space \( X \), the generating functional is defined as

\[
\Phi[\eta] = \prod \eta(x) = \exp \int \ln \eta(x) N(dx).
\]

In our case it will be more convenient to make use of the functions \( \eta(x) \), of the type \( 1 + \xi(x) \) or \( 1 - I(R) + \xi(x) \), where \( \xi(x) \) is a bounded function which is identically equal to zero outside some closed volume (in the second case, outside the interval \( I(R) \)) and \( I(R) \) is an indicator function which is equal to unity in the domain \( R \) and is equal to zero outside it. Then we can write the functional as follows [4, 5]:

\[
\Phi[1 + \xi] = 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \int \xi(x_1) \ldots \xi(x_k) m_k(dx^k),
\]

\[
\Phi[1 - I(R) + \xi] = \sum_{k=1}^{\infty} \frac{1}{k!} \int \xi(x_1) \ldots \xi(x_k) p_k^{(\xi)}(dx_k).
\]

The functions \( \xi \) should be understood as the k-fold integration.

The functions \( c_k \) (factorial semi-invariants) as well as \( m_k \) characterize the local probability features of the process, but in contrast to these, the functions \( g_k(dx^k) \), when \( k > 2 \), are equal to zero when the stochastic interconnection between the phenomena vanishes. The functions \( q_k^{(n)} \) (called "cluster functions") characterize the tendency of the phenomena to combine themselves into clusters containing \( n \) elements; a positions value thereof indicates a more frequent than random occurrence of such clusters in the process and vice versa.

For a Poisson process where there are no stochastic connections between the events, we have

\[
p_{\lambda}^{(n)}(A_1) = \frac{[\lambda(A_1)]^n}{n!} e^{-\lambda(A_1)}, \quad m_k(A) = [\lambda(A)]^k,
\]

\[
c_k(A) = \lambda(A), \quad c_1 = c_2 = \ldots = 0,
\]

\[
q_k^{(n)} = q_k^{(n)}(A) = \lambda(A), \quad q_k^{(n)} - q_{k-1}^{(n)} = \ldots = 0,
\]

\[
\Phi[1 + \xi] = \exp \sum \int \xi(x) \lambda(dx).
\]

In these formulas \( \lambda(A) \) is equal to the mean number of phenomena in the domain \( A \).

The functions \( c_k \) and \( q_k^{(n)} \) can take both positive and negative values, in contrast to the functions \( m_k \) and \( p_k^{(n)} \), which, by definition, are non-negative. They can be calculated if we know the functions \( m_k \) and \( p_k^{(n)} \) of the k-th (n-th) or lower orders [5, 8]. For instance

\[
c_1(dx_1) = m_1(dx_1),
\]

\[
c_1(dx_1 \times dx_2) = m_1(dx_1 \times dx_2) - m_1(dx_1) m_1(dx_2).
\]

The functions \( c_k \) and \( q_k^{(n)} \) are connected with one another by relationships which are completely analogous to the formulas (3) and (4); since \( c_0 = 0 \)

\[
\sum_{k=0}^{\infty} q_k^{(n)}(R) = 0.
\]

All the functions listed above and the relationships between them can be defined in terms of the variational differentiation of the functional [4]. For example

\[
n! p_k^{(n)}(A_1 \times \ldots \times A_n) = \delta_k^n \Phi[1 - I(R)] = \left\{ \frac{\delta^n}{\delta \theta_1 \ldots \delta \theta_n} \Phi \left[ 1 - I(R) + \sum_{i=1}^n \theta_i \xi_i(x) \right] \right\}_{\theta_1 = \ldots = \theta_n = 0},
\]

where \( \xi_i(x) = I(A_i) \) and \( \theta \) is a real variable so that \( 1 > \theta > 0 \). While differentiating the latter expression, we should bear in mind that

\[
\Phi \left[ 1 - I(R) + \sum_{i=1}^n \theta_i \xi_i(x) \right] = \sum_{k=1}^\infty \sum_{n=0}^k \frac{n!}{k_1! \ldots k_n!} \int \xi_1(x_1) \ldots \xi_n(x_n) \xi_{n+1}(x_{n+1}) \ldots \xi_l(x_l) p_k^{(n)}(dx^k),
\]

where \( l = \sum_{i=1}^n k_i \). For the functions \( c_k \), we have

\[
c_k(A_1 \times \ldots \times A_n) = \delta_k^n \Psi[1]
\]

where \( \Psi[\eta] = \ln \Phi[\eta] \).
We shall now examine the case when, in some part $A$ of the domain $R$, the position of all the points (events) is known. Then, in the remaining part of the domain $B$ we can determine the conventional probability of occurrence of the events:

$$P(N(B) = 0, N(dy_1) = 1, \ldots, N(dy_n) = 1 | N(A) = 0, N(dx_1) = 1, \ldots, N(dx_m) = 1)$$

or

$$P(N(A) = 0, N(dy_1) = 1, \ldots, N(dy_n) = 1 | N(B) = 0, N(dx_1) = 1, \ldots, N(dx_m) = 1) = 1$$

(16)

Multiplying (17) by $\prod_{n=1}^{\infty} \zeta(y^n)$ (the function $\zeta(y)$ is different from zero only in the domain $B$), integrating with respect to $y^n$ and adding, we get [9]

$$\sum_{n=1}^{\infty} \int \zeta(y_1) \ldots \zeta(y_n) p_{x}^{(n)} (dy^n) \frac{(n + m)!}{n!} p_{x}^{(n)} (dy^n) =$$

$$= \frac{1}{p_{x}^{(n)} (dy^n)}$$

(18)

or, as can be proved easily,

$$\frac{\Phi[1 - I(R) + \zeta | x^2]}{\Phi[1 - I(A) + \zeta]} =$$

$$\frac{\delta_{\zeta} \Phi[1 - I(R) + \zeta]}{\delta_{\zeta} \Phi[1 - I(A) + \zeta]}$$

(19)

Here the function $\zeta(x) = I(dx)$. Knowing the expression for the conventional functional in the domain $B$, one can obtain any conventional statistical characteristics. Thus, the mean number of phenomena in the interval $dy$ of the domain $B$, is equal to

$$m_c(dy | x^2) = c_1(dy | x^2) = \delta_1 \ln \Phi[1 | x^2] - \delta_1 \ln \delta_1 \Phi[1 - I(A)]$$

(20)

In this formula, $\zeta(y) = I(dy)$.

Let us assume, for the sake of illustration, that two phenomena occur in the domain $A$ at the points $x_1$ and $x_2$. Then,

$$c_1(dy | x^2) = \delta_1 \Psi =$$

$$= \delta_{2 \zeta} \Psi + \delta_{\zeta \zeta} \Psi + \delta_{\zeta \zeta} \Psi + \delta_{\zeta \zeta} \Psi + \delta_{\zeta \zeta} \Psi.$$

(21)

In this formula $\Psi$ denotes $\Phi[1 - I(R) + I(B) + \zeta + \zeta + \zeta = \ln \Phi[1 - I(A) + \zeta + \zeta + \zeta + \zeta].$

Representing the functional $\Phi$ in the form of (9) and using the analogous expression (14), we can easily demonstrate that

$$\delta_1 \Psi = \sum_{m=1}^{\infty} m q_x^{(m)} (dy \times B^{m-1}),$$

$$\delta_{2 \zeta \zeta} \Psi = \sum_{m=1}^{\infty} (m - 1)(m - 2) q_x^{(m)} (dx_1 \times dx_2 \times dy \times B^{m-4}),$$

$$\delta_{\zeta \zeta \zeta} \Psi = \sum_{m=1}^{\infty} m(q_x^{(m)} (dx_1 \times dx_2 \times B^{m-4}).$$

(22)

Thus, the first expression in (21) takes into account the event-clusters, none of which fall in the interval $A$. The first expression in the numerator of the fraction characterizes the "contribution" of those clusters composed of three or more members, in which one event is located at the point $x_1$, the second at $x_2$, the third at $y$ and the remaining events in the domain $B$ (these can fall in the interval dy also). The explanation of the remaining expressions in the formula (21) is obvious.

Proposition on the basis of formula (20), in sum, reduces itself to a breaking up of the aggregate of events in $A$ into individual groups in all possible ways and to an assessment of the possibility of one or more of the elements of this group being located in the interval dy.

2. PROBABILISTIC MODELS OF A SEISMIC REGIME

It is clear from the preceding section that the problem of the stochastic description of a seismic regime could have been solved in principle if we had succeeded in determining the general form of the generating functional. The functional is expressed in terms of an infinite sum of probability functions and hence, as a first step, we can try to express it in terms of a finite sum of functions. Here the expressions (7) and (9) come in most handy. Indeed, if we equate the terms with $k \geq m$ or $n \geq m$ to zero in the expressions (6) and (8), it will mean that in the domain $R$ there cannot be more than "$m$" events - an assumption which is clearly improbable for large domains of $R$. An analogous truncating of the series for the expressions (7) and (9) means that the clusters of events of "$m$" or more elements are considered to be simply random collection from lower order groups.

If we neglect all the terms of the series except the first, the resultant process will be a Poisson process (see formulas (10)). As is well known, the Poisson process is quite often invoked for the description of an aggregate of earthquakes [1, 3], but detailed investigations have, as a rule, led to the strengthening of the Poisson hypothesis, i.e., to the purely random character of the seismic regime. The point process, in which only the first two terms in (7) are different from zero, is discussed in detail in [10]. There this process is called a Gauss-Poisson process. In that work it is shown that not all the functions of the factorial semi-invariants possess a probability solution, i.e., for certain
functions \( \phi(q) \) - it is possible to obtain negative values for the functions \( p_n(\theta) \), which is impossible. This is, to some extent, analogous to the situation in statistical hydrodynamics, where a similar problem arises when the series in the characteristic function of turbulent flow is truncated [11]. A deficiency which is still more serious, is the lack of accounting for large clusters. Since we are interested only in very intense seismic phenomena and since these are statistically linked with large event-clusters (foreshocks and aftershocks), the approach described above will only be of very limited utility for a probabilistic description of the seismic regime.

One can simplify the problem to a great extent if one assumes that the seismic regime is stationary in time as well as homogeneous and isotropic in space. For earthquakes it is quite possible that the stationary hypothesis is fairly strictly observed, since it is difficult to expect a significant change in the earthquake-causing tectonic bodies in a brief interval of time in the geological scale. As for homogeneity, it is quite evident that large seismic phenomena which affect an appreciable part of a seismic region, occur in an essentially inhomogeneous medium. For weaker phenomena one can obviously consider the conditions to be fairly homogeneous in a number of cases. We notice again an analogy with statistical hydromechanics, where large vortices are affected by the inhomogeneity of the medium while it can be neglected [11] for the small-scale turbulence of the flow.

For stationary and homogeneous processes the formulas for the factorial semi-invariants become considerably simpler [5]. The first semi-invariant reduces itself to the "frequency law of earthquakes" [7] or, after suitable normalization, to the energy-wise distribution of the seismic phenomena. The second semi-invariant, in this case, is a function of the four variables: \( r, r, k_1, k_2 \), where \( r \) is the distance between the hypocenters of the phenomena, \( \tau \) is the time difference of their occurrences, \( k_1 \), and \( k_2 \) are the energy classes of the phenomena (\( K = \lg E \)).

In the real physical systems the following ergodicity condition is usually satisfied:

\[
\lim_{|A|\to\infty} \frac{1}{|A|} \sum_{x \in A} \phi(e^{i2\pi k_1 x_1}) = \phi(0),
\]

Here \( |A| \) denotes the volume of the domain \( A \) in \( K_1 \times [k_2, +\infty) \), where \( K_1 \) is the interval in the space \( U = H \times T \) and \( T \) is the energy interval \( [K_2, +\infty) \) is taken from class \( K_2 \) upward. The condition (23) means that the statistical interrelation between the events becomes weak with an increase of the distance, the weakening being so rapid as to ensure the convergence of the integral. Since the cluster functions are uniquely connected with \( k_2 \), one should expect that the functions \( \frac{1}{|A|} q_{\Lambda_n}(R) \) also will tend to some finite limit when \( |A| \to \infty \). With the expansion of the domain \( A \) the total process will be composed of a large number of almost mutually independent sub-processes, corresponding to some break-up of the domain \( A \). In this case, according to the probability theory, the probability distribution of the total process must converge to an infinitely divisible distribution. The general form of the generating function of the infinitely divisible distribution for integer random variables (complex Poisson distribution) is defined by the formula (see [2] and [12], ch. XII):

\[
\phi(q) = \exp \left( \sum_{n=1}^{\infty} \lambda_n q^n \right),
\]

where \( \lambda_n = \sum_{i=1}^{\infty} \lambda_i \) and \( \lambda_i \geq 0 \) when \( i \geq 1 \). From the formulas (9), (12) and (24) we see that the functions

\[
\frac{q^{(\alpha)}(R)}{q^{(\alpha)}(\Lambda_n)} = \Lambda_n
\]

should be interpreted as the probability of the cluster from \( n \) elements in the total cluster structure of the process [12].

These ideas are confirmed by the dispersion measurements relating the number of seismic phenomena (see for example [1, 3]). The magnitude of the dispersion is usually larger (especially for the crustal earthquakes) that the value which is characteristic for the Poisson process of the same intensity. Such a relationship is a typical feature of the clustering of events, when the phenomenon realized generally exceeds the probability of the other events in its immediate or distant environs.

The so-called Poisson cluster process [5, 4], in which the event-clusters are distributed near the cluster center in accordance with some probability distribution law (independent for each cluster), serves as an adequate apparatus for the description of clustering point systems. The process of the cluster centers is, in this case, a Poisson process. Essentially, the seismologists have, for a long time, being using the similarity principles, since it is well known that when the aftershocks (i.e., cluster phenomena) are discarded, the new seismic process becomes much more akin to a Poisson process than the primary process. We should note here that a similar description of the seismic regime is generally an approximate one, since we can only partly take into account (with the assigned structure of the cluster) the possible "mutual repulsion" between the phenomena (for instance, due to the complete or partial "draining" of the deformation energy in the focus). The independence of the event-clusters leads to the result that two phenomena from different clusters can practically be superposed one over the other. However, a change in this condition, brought about by some form of dependence, makes the theory of the process so complex that it becomes probably unworkable. Hence, it is best to examine the Poisson cluster process as some idealization of the real physical picture.

It will be most expedient if we take the center of the seismic event cluster to be the most intense phenomenon itself, even though, in principle, it is possible to develop a theory for the case when the cluster is characterized by some other parameters [3]. With regard to the probabilistic model of the interrelation between the members of cluster, the best developed ones at the present time are the two models which are called the Bartlett--Louis and Neuman--Scott models in [3]. Figure 1 shows the graphs of these processes, the peaks corresponding to the seismic phenomena and the arcs, to the stochastic interrelations between them. It is evident that the description of the seismic regime with the help of these processes will be generally unsatisfactory, since none of them permits us to take into account the secondary clustering of the events close to the intense phenomena of the cluster (strong aftershocks and foreshocks of the main phenomena). Hence, as a model of the statistical interrelation between the phenomena in the cluster, one is best advised to take the branching process, which is schematically presented at Fig. 1c.

![Fig. 1. Graphs of the processes:](image)
The total process can thus be described in the following way: in the first place, following [1], let us assume that the seismic regime is composed of mutually independent primary phenomena, distributed according to some function $p(dx)$ and stochastically connected with the primary cluster phenomena. Every one of the cluster phenomena can also be the "parent" of its own subgroup, etc. Henceforth we shall call the parent of the group or subgroup the primary phenomenon and the member of the corresponding subgroup the secondary phenomenon. Secondly, every primary phenomenon "generates" the secondary one irrespective of its previous history and the position of the other phenomena. Thirdly, the probability of occurrence of the secondary phenomenon in some energy region, is $p(dK)$ and the distribution function of its position, when it occurs at the point $K$, is $f(A, Kv)$. Here $A$ is the interval in the four-dimensional space $U=HxT$ and $v$ represents the coordinate of the primary phenomenon.

The above-listed conditions define some random branching process viz. a "production process with immigration" [13]. The role of the "immigrants" (parents of independent families) is played here by the main seismic phenomena. In contrast to a number of other applications of branching process the branching takes place, in this case, not in time but in energy. Every secondary phenomenon is weaker than the primary from the energy standpoint, while in time the former can occur later (afterershock) or earlier (foreshock) than the primary phenomenon. The position of the cluster members will, in this case, be determined by the conventional branching process (provided the main phenomenon belongs to the class $L$ and has occurred at the point $v$). For this process, if the functions $f(dK)$ and $f(A, Kv)$ are known, we can obtain any probability functions, including the conventional factorial moments of the number of phenomena of class $K$ and above: $M_K(Av, L)$ and $M_K^* (Av, L)$ [14, 15].

The generating functional of the complete process, i.e., the one consisting of the main and the cluster phenomena, in the general case, can be written in the following way [4, 13]:

$$\Phi_{\alpha}[v] = \exp \left[ \int [G_{\alpha}[v, L] - 1] v(dL \times dv) \right],$$

where $G_{\alpha}(L, v)$ is the conventional functional of the cluster members with the logarithm of their energy greater than $K$. From this we get, for example, an expression for the first two factorial semi-invariants:

$$C_2(A) = \int \int M_x(A, v, L) v(dL \times dv),$$

$$C_{2x}(A, x) = \int \int M_{x,x}(A, x, v, L) v(dL \times dv).$$

The above-mentioned conditions of the branching model permit one to determine, in a comparatively simple way, the functions $q_n^{(0)}$ for the expansion (9) of the complete functional of the process. It is convenient to take the domain of $R$ again in the form $A \times [K, +\infty)$. Then,

$$q_n^{(0)}(dx^{n+1}) = I(R | y) v(dy) \frac{1}{n!} \sum_{m=0}^{n} \frac{1}{m!} \sum_{m=0}^{n} Q_m^{(0)}(dx^{n+1} \times R^{m+1} | y, w) v(dy).$$

In the formula, $I(R | y) = 1$ if the main event occurs in the interval $R$ and is equal to zero in the opposite case; $R = X - R$, and $Q_m^{(0)}(dx^{m}, ly, w)$ is the probability that there are exactly $m$ members of the cluster, belonging to class $K$ or a higher class in the states $dx_1 \ldots dx_m$, on condition that the main phenomenon occurred at the point $y$ and the structure of the family (i.e., the scheme of interrelationships between the primary and the secondary phenomena in the cluster) is described by the random tree $w$[15]. For $n$ members of the cluster there exist $n!$ such trees. As can be seen from formula (28),

$$\lim_{|x| \to \infty} q_n^{(0)}(dx^{n+1}) = \frac{1}{n!} \sum_{m=0}^{n} \frac{1}{m!} \sum_{m=0}^{n} Q_m^{(0)}(dx^{n+1} \times R^{m+1} | y, w) v(dy).$$

The function $Q_n^{(0)}$ can be determined, proceeding from the conditions of the branching process,

$$Q_n^{(0)}(dx^{n+1} | y, w) = \exp -\sum_{m=0}^{n} \mu(K_m + \infty) \prod_{i=0}^{n} f(dA_i, K_i) v(dK_i).$$

The product on the right hand side of (30) is also taken over the intersections of the primary ($v$) and the secondary ($w$) phenomena, corresponding to the given tree.

It is interesting to study the behavior of the process over intervals many times larger than the mean dimension of the clusters, i.e., when it is possible to discard the space-time differentiation of the cluster members. For the sake of simplicity we shall assume that

$$\mu(dK) = \mu \cdot dK = \text{const} \cdot dK, \quad v(dL) = v \cdot dL = \text{const} \cdot dL$$

for $K < K_\alpha$.

In this case the conventional generating function of the number of the cluster members, including the main phenomenon, can be written as follows [13]:

$$L(s) = \exp -\mu(L - K)$$

The conventional distribution of the total number of the cluster members can be obtained through an $n$-fold integration of the expression (30) or through differentiating the expression (32) $(n + 1)$ times at the point $s = 0$.

The additional differentiation is necessitated by the fact that in (32) the main phenomenon is taken into account, whereas in formula (30) it is not. We have

$$Q_n^{(0)}(U^n | L) = e^{-\mu s L^n} \left(1 - e^{-\mu s L^n} \right)^n,$$

i.e., a geometrical distribution [12]. This distribution was employed in [16] for the approximation of the number of the subsequent earthquake shocks for a fixed difference $L - K$.

The generating functional of the total number of phenomena will have the form [13]:

$$q_s(z) = \exp \left[ \int [g_s(s | L) - 1] v(dL) \right]$$

$$= \exp \left[ \frac{\ln(1 - s + s \exp -\mu(K) - 1)}{-\mu(K)} \right] v(K) - K)$$

Expression (35) shows that the distribution of the number of phenomena in large volumes will follow a negative-binomial distribution. This distribution is a fairly good approximation of, for example, the number of seismonaocoustic impulses in unit time (see [17] ch. III). The form of the latter expression in the formula (34) bears out the fact that the distribution of the cluster number will be logarithmic in this case (see [12], ch. XII and [8]), with the probability of the clusters from $n$ members being equal to (when $n > 1$)

$$\Lambda_n(K) = \frac{1}{n!} \exp -\mu(K) - K)$$

The mean dimension of the cluster is given by

$$M(K) = \frac{\exp -\mu(K) - K)}{\mu(K) - K)}.$$
We see from formulas (36) and (37) that the degree of clustering should increase with the decrease of K. In references 1, 18 a somewhat different distribution is given which is employed, with success, for the approximation of the distribution of earthquake clusters. In these references the quantity $\Lambda_n$ is taken to be proportional to $n^\beta$. However, the logarithmic distribution for a correct choice of parameters will be little distinguished from this distribution, especially when the limits of the variations of n are small, as can be verified easily.

Differentiating (35) with respect to s at the point s = 1 we get for the first semi-invariant of the total process,

$$C_s = -\frac{\mu}{n} \exp \left( \mu (K_n - K) - 1 \right).$$

Differentiating in its turn the expression (38) with respect to K in order to obtain the distribution density, and taking logarithms, we obtain the frequency law of the earthquakes in the customary linear form $[1, 7]$. In other words, the energy distribution of seismic phenomena will follow Pareto's law (see [17] ch. IV). It is also evident that, for the model (31), the quantity $v$ is proportional to the seismic activity and the dividing coefficient $\mu$ is equal to the parameter $\gamma$ of the energy distribution of seismic phenomena with an accuracy up to the multiplier in 10.

Thus, the suggested branching model permits one to solve the problem of computation of the higher order moment functions in the expansions of (7) and (9). These functions can be used for a probabilistic description of the seismic regime and for its extrapolation from formulas (19)-(22). A similar computation can be carried out by the Monte-Carlo method. The obtained calculations show that the model approximates well the earthquake distributions studied so far, one makes the simplest possible assumptions regarding the form of the probability functions defining the model.

3. POSSIBILITIES OF A STATISTICAL DETERMINATION OF THE CHARACTERISTICS OF THE SEISMIC REGIME.

We shall now examine the question of the type of characteristics one can study from the available realizations of the random process, i.e., from the observed aggregate of seismic phenomena. These characteristics should naturally satisfy two conditions: (1) a comparative simplicity in the calculation within the framework of the assumed process model; (2) a high degree of generality so that these characteristics can be compared for different conditions and regions.

In the light of the above-mentioned criteria, it is clear that not all the functions mentioned in the preceding sections are uniformly suitable for a statistical study. While investigating the function $p_d(r)$ and $q_d(r)$, it is necessary to determine the domain R by some rational method or other -- a task which is quite difficult. The local functions $m_k$ and $q_k$ are, however, free of these difficulties, and since the factorial semi-invariant can be easily expressed in terms of the same or lower order moments, the problem boils down to a study of factorial moment functions.

For a statistical study of the higher order moment functions, it is expedient to express them in terms of the conventional lower order functions. In the expression (19), if we take the domain $A = \sum dx_i$, it is easy to obtain

$$m_n(dy^m | x_i) = \frac{m_{n+m}(dx^m \times dy^m)}{m_n(dx^m)}. \quad (39)$$

From this follows, for example,

$$m_n(dx^m \times dx_2) - m_n(dx_2) - m_n(dx^m | x_2). \quad (40)$$

The function $m(dx^m | x_2)$ corresponds to the "intensity function" referred to in [19] and it characterizes the mean number of phenomena in the interval $dx^m$ on condition that the other phenomenon occurred in $x_2$. For the homogeneous and stationary case, this function depends only on the energy of the phenomena $(E_1, E_2)$ and the distance between the phenomena $(\tau, E_2)$ of all possible intervals between the events and to a corresponding normalization of the statistics [19].

A study of the higher order moments will be naturally fraught with difficulties mainly due to the dependence of higher moments on a large number of variables: even the third moment, in the stationary and homogeneous case, depends on 8 variables. This multidimensionality renders the choice of functions for the approximation of statistical data quite difficult. We can try to overcome this difficulty by investigating the integral characteristics, such as $S_2(K)$ in formulas (23). A quantity analogous to $S_2$ was studied for example, in [3]. If one could succeed in getting at least some values of $S_2(K)$, it would have been possible to make an inference on the cluster structure of the process, i.e., to determine the coefficients $\Lambda_n$. In a similar study of the asymptotic characteristics of semi-invariants we should bear in mind the fact that the widening of the domain R in space usually appears impossible on account of the intrinsic inhomogeneity of the process. Obviously, a study of the behavior of these quantities for increased time intervals will be more promising.

If the characteristics of the process listed above are subjected to a fairly thorough statistical analysis, it will be possible to compare them with the theoretical characteristics which are assumed for some probabilistic model of the process. By such a comparison one can compute the parameters of the model and thus determine the probabilistic structure of the entire process. For

![Fig. 2. Distribution of the intervals between seismo-acoustic impulses:](image)
the branching model stipulated earlier it is necessary for example, to determine the branching parameter $\mu$, the function $v(x)$ and the distribution function $f(A, K|Y)$.

The last function can be given in the form of some sufficiently simple function of $\frac{\alpha}{\mu}$ and $\frac{\beta}{\mu}$, where $\alpha$ and $\beta$ are the space and time distances between the primary and the secondary phenomena and the expressions at the denominator of the arguments are given from similarity considerations or automodeling of the regime.

For the sake of illustration, we shall now investigate the conventional moment $m(x)$ which, according to (40), defines the second factorial moment. We have used, as basic data, the materials of the multichannel recording of seisimo-acoustic impulses, generated by the destruction of a coal seam under the action of forces due to rock pressure. Details concerning the experimental technique and processing method are contained in [17] (ch. IV). Figure 2a, b depict the non-normalized distribution of the number of impulses with energy $E_2$, incident before (left-hand side of the graph) or after the impulse with energy $E_1$. Naturally, when $E_1 = E_2$ the graph is symmetrical and hence only its right half is given. The upper graphs in Figs. a and b correspond to the distance between the foci of the impulses ($r$) less than 2 m, the middle ones -- 2 to 4 m and the lower ones -- more than 4 m. This distance, which is associated with the small thickness of the seam (0.5 m), was taken in the plane of the seam, i.e., the initial spatial distribution was considered to be two-dimensional. The emergence of strong impulses ($E > 1$ joule) is clearly statistically intercon-

For a small distance $r < 2$ m, the emergence of another impulse in the first 20-30 sec in the wake of one impulse is very probable. When the distance is $2 < r < 4$ m, the time of the maximum is shifted by 20-40 sec and when the distance is more than 4 m, the shift is by 40-60 sec. The cause of this may be the "destruction wave", studied in [17] (ch. IV) by somewhat different methods. Its velocity works out to be about 0.1 m/sec. An analogous migration of the hypo-

center of earthquakes was also seen by seismologists (see, for example [20, 21]).

For weaker impulses (Fig. 2b), we see some increase of their number immediately before and after a strong impulse, but this effect cannot be localized in the spatial sense. Possibly, the mechanism of mutual connection between the phenomena is somewhat different here. Another reason for the difference between the Figs. 2a and b may be the small determination accuracy of the focal coordinates of the weak impulses ($E < 1$ joule): the root-mean-square error is 1.5-2.5 m in this case, i.e., it is comparable with the variation limits of $r$.

The large inhomogeneity and nonstationarity of the process of destruction of the seam in the mine can also introduce serious errors.

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