Point sources of elastic deformation: elementary sources, dynamic displacements

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Summary. We construct a catalogue of all the possible elementary point sources of seismic waves. There are three general classes of sources, two spheroidal and one toroidal. We consider excitation functions for these point-like sources as well as for sources of finite size in far-, intermediate- and near-field for an infinite homogeneous isotropic medium. The sources corresponding to seismic-moment tensors for the second-, third- and fourth-ranks are considered in more detail; we identify 10 different seismic sources in this range: one monopole, two or three dipoles, three quadrupoles, etc. For the step-function of the scalar seismic-moment release, the amplitude spectrum for the third-rank sources is proportional to the angular frequency \( \omega \) in the region below the corner frequency \( \omega_{cr} \). The fourth-rank sources have an \( \omega^2 \) spectrum in the same range. The possibility of separate and simultaneous inversion of seismic body-wave data and static deformation data for sources of different order is discussed. Some equivalent-force moment higher-rank sources are 'shielded' by lower-rank sources of the same order; the former sources cannot be inverted from seismic data without additional assumptions. Because of their simple radiation pattern, the lower order multipoles, i.e. the monopole and dipoles, are the first sources other than the double-couple which should be considered for inversion.

Key words: elastic wave sources, higher-rank seismic-moment tensors, inversion, vector multipole expansions

1 Introduction

This paper continues our study of elementary point sources of elastic deformation. In the first paper of this series (Kagan 1987a), we considered general principles for classifying elastic sources and, in particular, static deformation caused by these sources. For purposes of brevity we will refer to that paper as H1. To distinguish equations and figures of this paper from those of H1 and other references, we use upper-case letters (e.g. eq. 1 in H1 vs Eq. 1 here). We use the notation of H1, supplementing it when necessary. As in H1, through-
out this study we use the term rank, denoted by $L$, for reference to the Cartesian tensors; the term order, denoted by $l$, will be used only for spherical harmonics or for weight of the tensor representations of the 3-D rotation group.

In the present study we extend our results primarily to dynamic deformation or to seismic waves produced by elementary elastic sources. We will present our conclusions in two mathematical formalisms as we did in H1: vector spherical harmonics and symmetric trace-free (STF) tensors (Thorne 1980). Our reason for using both the multipole-moment and the STF tensor (STFT) apparatus is the same as in H1: it allows a consistent and formal treatment of source properties, and gives us an opportunity to use both vector spherical harmonics and seismic-moment tensors to describe sources of elastic deformation.

The patterns of static displacement differ significantly from the excitation of seismic waves by different elastic sources. As was mentioned in H1, the sources of dynamic displacement are tensor functions of frequency or time, whereas the static sources can be represented by tensors. Thus we need to explore the dependence of seismic-wave deformation on frequency (or time) and on the distance between a source and a receiver. Unlike sources of static deformation, there are no null classes of the STFT sources of elastic waves, so the list of dynamic sources is larger and more complex. As the frequency of elastic waves goes to zero, the dynamic sources should transform themselves into static sources; this process will be described later in the paper. We discuss the properties of elastic deformation solutions both in time and frequency domains, as well as the properties of seismic waves in near- and far-fields.

There are many studies of seismic radiation from sources described by the first- and second-rank seismic-moment tensors (Morse & Feshbach 1953; Ben-Menahem & Singh 1968; Randall 1972; Phinney & Burridge 1973; Gilbert & Dziewonski 1975; Bache & Harkrider 1976; Petrovitch 1978, section 15; Cormier 1980; Aki & Richards 1980; Ben-Menahem & Singh 1981). Sources of elastic waves other than second-rank tensors, i.e. the standard double-couple and explosion sources, are now of interest to researchers (Doornbos 1982; Kanamori & Given 1982; Stump & Johnson 1982; Silver 1983; Dziewonski & Woodhouse 1983; Eissler & Kanamori 1987). Here we are interested mainly in describing seismic waves generated by the higher-rank sources, although our general formulas include the above-mentioned second-rank sources as specific cases. As shown in H1, higher-rank tensors may produce displacements of low-order $l$; for example, there are three indigenous dynamic sources of dipolar radiation. Literature sites several solutions of elastic radiation from various sources (Archambeau 1968; Snoke 1976; Stevens 1980; Barker & Minster 1980; Stevens & Day 1985; Glenn et al. 1986; see also references above); usually these solutions are specific to the problem they resolve. In this study we wish to describe a general and complete category of sources, such that deformation data can be inverted without commitment to a particular model of the elastic source.

The results presented here may be used for inversion of source properties from records of seismic waves caused by earthquakes and explosions. The inversions may be especially important for sources of deep earthquakes (Randall 1968), where the only information available is usually that of the seismic radiation records. We show that in a dynamic case much more information can be extracted from deformation data than in a static case; combining the data from both cases allows us to study sources which cannot be investigated by using only one type of data. Some of the inversion results could also be used for discriminating between earthquakes and explosive sources (Stevens & Day 1985; Evernden, Archambeau & Cranswick 1986).

As we mentioned in H1, the only information to be retrieved from elastic deformation data, static or dynamic, is a set of multipoles or a set of the STF source tensors; these
tensors are essentially the multipoles written in a different notation. The STF source tensors corresponding to higher-rank seismic-moment tensors demonstrate a complexity of the earthquake focal mechanism (this will be discussed in detail in our next paper). Similarly, the presence of higher-rank tensors in a source description of an explosion indicates complex prestress patterns near the explosion point (Stevens 1980).

2 Dynamic Green's function and its expansion in vector spherical harmonics

The dynamic equation of elasticity in an infinite, isotropic, homogeneous medium (the dynamic Navier equation) is

\[(\lambda + 2\mu) \text{grad} \, \text{div} \, \mathbf{u} - \mu \text{curl} \, \text{curl} \, \mathbf{u} + \rho \omega^2 \mathbf{u} = 0,\]

or

\[(\lambda + \mu) \text{grad} \, \text{div} \, \mathbf{u} + \mu \nabla^2 \mathbf{u} + \rho \omega^2 \mathbf{u} = 0, \tag{2.1}\]

where \(\mathbf{u}\) is the displacement amplitude, \(\lambda\) and \(\mu\) are Lamé's elastic constants, \(\rho\) is the density of a medium, and \(\omega\) is the angular frequency.

The Green's function expansion for the Navier equation is

\[G(\omega) = i\omega \left[ \frac{1}{\alpha(\lambda + 2\mu)} \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} L_{lm}^l L_{lm}^l + \frac{1}{\beta\mu} \sum_{l=1}^{\infty} \sum_{l=-l}^{l=l} (M_{lm}^l N_{lm}^l + N_{lm}^l M_{lm}^l) \right] \tag{2.2}\]

(Morse & Feshbach 1953, p. 1875; Ben-Menahem & Singh 1968, 1981, eq. 4.172), where \(i = \sqrt{-1}\), and \(G\) is a second-rank tensor. The plus sign in the subscript corresponds to the internal solution, the minus sign is for the external part of wave radiation. \(\alpha\) and \(\beta\) are velocities of longitudinal and transverse waves, respectively

\[\alpha = \frac{\sqrt{\lambda + 2\mu}}{\rho}, \quad \text{and} \quad \beta = \frac{\sqrt{\mu}}{\rho}. \tag{2.3a}\]

The Hansen vectors \(L_{lm}^l\) in (2.2) depend on \(\Lambda_\alpha\), the vectors \(M_{lm}^l\), and \(N_{lm}^l\) have \(\Lambda_\beta\). By \(\Lambda_\alpha\) and \(\Lambda_\beta\), we denote reduced wavelengths of compressional and transverse waves, respectively:

\[\Lambda_\alpha = \frac{\alpha}{\omega}, \quad \text{and} \quad \Lambda_\beta = \frac{\beta}{\omega}. \tag{2.3b}\]

The Hansen vectors are

\[L_{lm}^l = \frac{1}{\sqrt{2l+1}} \cdot \left[ \sqrt{l} j_{l-1}(\Xi) Y_{l-1,l}^{l-1,lm^*} + \sqrt{l+1} j_{l+1}(\Xi) Y_{l+1,l}^{l+1,lm^*} \right], \tag{2.4a}\]

\[M_{lm}^l = j_l(\Xi) Y_{l}^{l,lm^*}, \tag{2.4b}\]

\[N_{lm}^l = \frac{1}{\sqrt{2l+1}} \cdot \left[ \sqrt{l+1} j_{l-1}(\Xi) Y_{l-1,l}^{l-1,lm^*} - \sqrt{l} j_{l+1}(\Xi) Y_{l+1,l}^{l+1,lm^*} \right], \tag{2.4c}\]

where \(\Xi = R/\Lambda\) and \(\xi = r/\Lambda\); \(r = (r, \theta, \phi)\) is a distance vector for a region outside the focal zone and \(R = (R, \theta, \phi)\) denotes a distance for a source region; \(Y_{l}^{l,lm^*}\) are the pure-orbital vector spherical harmonics (see Appendix A). Above

\(l'\) is \(l - 1, l\) or \(l + 1\). \tag{2.5}
We call
\[ n = l' - l + 1 \]  
the class of a source. The vectors \( L'^m \), \( M'^m \), and \( N'^m \) are obtained from (2.4) by replacing \( f'_t(\xi) \cdot Y'^{l',m'} \) with \( h'_t(\xi) \cdot Y'^{l',m} \). The function \( j'_t(\xi) \) in (2.4) is the spherical Bessel function of the first kind, \( h'_t(\xi) \) is the spherical Hankel function (Abramovitz & Stegun 1972, pp. 437–439; Morse & Feshbach 1953):
\[ h'_t(\xi) = j'_t(\xi) \pm i \cdot y'_t(\xi). \]  
The plus sign is used for ‘outgoing’ waves, the minus sign for an ‘incoming’ solution, and \( y'_t(\xi) \) is the spherical Bessel function of the second kind. Two formulas for the spherical Hankel function are useful
\[ h'_t(\xi) = \exp (i \xi) \sum_{k=0}^{\infty} \frac{(-i)^{l+k+1} (l+k)!}{2^{k} k!(l-k)!} \cdot \frac{1}{\xi^{k+1}}, \quad (2.8a) \]
and
\[ h'_t(\xi) = \sum_{k=0}^{\infty} \frac{(-1)^{k} k^{l+2} k}{2^{k} k!(2l+2k+1)!} - i \sum_{k=0}^{\infty} \frac{(-1)^{k} (2l-1)!!}{2^{k} k!(1-2l)(3-2l) \ldots (2k-1-2l)} \cdot \frac{1}{\xi^{l+1-2k}}, \quad (2.8b) \]
(Thorne 1980; Ben-Menahem & Singh 1981, p. 975). Here
\[ (2l+1)!! = (2l+1)(2l-1) \ldots 3 \cdot 1. \]
Later we use only an ‘outgoing waves’ solution; we show the corresponding Hankel function in (2.8a). The expansion for the spherical Bessel function of the first kind, which corresponds to standing waves, is the first sum in (2.8b) equivalent to \( j'_t(\xi) \) in Eq. 2.7.

We may obtain the static Green’s tensor from the dynamic tensor by taking an appropriate approximation of the Bessel and the Hankel functions (2.7) when \( \omega \to 0 \). To illustrate, we subdivide the spheroidal part of the Green’s tensor (2.2)
\[ G'^m_{\text{spheroidal}} = i\omega \left[ \frac{1}{\alpha(\lambda + 2\mu)} \frac{1}{L'^m L'^{im} - \frac{1}{\beta\mu} N'^l N'^{lm}} \right] \]
\[ = G'^{(1)}_{\alpha} + G'^{(2)}_{\alpha} + G'^{(3)}_{\alpha} + G'^{(4)}_{\alpha} + G'^{(1)}_{\beta} + G'^{(2)}_{\beta} + G'^{(3)}_{\beta} + G'^{(4)}_{\beta}, \quad (2.9a) \]
where
\[ G'^{(1)}_{\alpha} \propto j_{l-1}(\xi_{\alpha}) \cdot Y'^{l-1,lm} \cdot h_{l-1}(\xi_{\alpha}) \cdot Y'^{l-1,lm}, \]
\[ G'^{(2)}_{\alpha} \propto \overline{j_{l-1}(\xi_{\alpha})} \cdot Y'^{l-1,lm} \cdot \overline{h_{l+1}(\xi_{\alpha})} \cdot Y'^{l+1,lm}, \]
\[ G'^{(3)}_{\alpha} \propto \overline{j_{l+1}(\xi_{\alpha})} \cdot Y'^{l+1,lm} \cdot \overline{h_{l-1}(\xi_{\alpha})} \cdot Y'^{l-1,lm}, \]
\[ G'^{(4)}_{\alpha} \propto j_{l+1}(\xi_{\alpha}) \cdot Y'^{l+1,lm} \cdot h_{l+1}(\xi_{\alpha}) \cdot Y'^{l+1,lm}, \quad (2.9b) \]
with similar expressions for \( G'^{(1)}_{\beta} \). As \( \omega \to 0 \), both \( G'^{(3)}_{\alpha} \) and \( G'^{(3)}_{\beta} \) tend to zero. The only difficulty arises when the sum of \( G'^{(2)}_{\beta} \) and \( G'^{(2)}_{\beta} \) is calculated. The products of the first terms of \( j_{l-1}(\xi) \) and \( h_{l+1}(\xi) \) of the expansion (2.8b) in the sum \( G'^{(2)}_{\alpha} + G'^{(2)}_{\beta} \) cancel, so we need to take second terms, \( k = 1 \), from Eq. (2.8b). The cancellation of the \( k = 0 \) terms, corresponds to near-cancellation of close-range, longitudinal and transverse waves (Wu & Ben-Menahem
1985). We then obtain two terms in the sum of $G^{(2)}_{\alpha} + G^{(2)}_{\beta}$, one of which depends on $R'/r'$, and the other depends on $R'/r' + 2$. The first of these terms and the sum $G^{(1)}_{\alpha} + G^{(1)}_{\beta}$ form the first series in the static Green's tensor expansion (eq. 2.2 in H1). The second term and $G^{(4)}_{\alpha} + G^{(4)}_{\beta}$ form the third series of the static tensor.

The STFT versions (see Appendix A and eqs B.4 of H1) of the Hansen vectors are

$$L_{im}^{\ell m} = \frac{l}{2l+1} \cdot \mathcal{V}_{j_{i-1}}^{\ell m} \cdot N_{A_{i-1}} \left[ j_{i-1}^{\ell m} (\Xi_\alpha) + j_{i+1}^{\ell m} (\Xi_\alpha) \right] - \mathcal{V}_{A_{i}}^{\ell m} \cdot N_{A_{i}} \cdot \mathcal{V}_{j_{i+1}}^{\ell m} (\Xi_\alpha)$$

$$= i \cdot \mathcal{V}_{j_{i-1}}^{\ell m} \cdot N_{A_{i-1}} \cdot \frac{j_{i}^{\ell m} (\Xi_\alpha)}{\Xi_\alpha} - \mathcal{V}_{A_{i}}^{\ell m} \cdot N_{A_{i}} \cdot \mathcal{V}_{j_{i+1}}^{\ell m} (\Xi_\alpha),$$

(2.10a)

$$M_{im}^{\ell m} = -i \cdot C_1 \cdot j_{i}^{\ell m} (\Xi_\beta) \epsilon_{j_{i}^{\ell m} \cdot \mathcal{V}_{q}^{\ell m} \cdot N_{A_{i-1}}.$$ (2.10b)

$$N_{im}^{\ell m} = C_3 \left[ \mathcal{V}_{j_{i-1}}^{\ell m} \cdot N_{A_{i-1}} \frac{l + 1}{2l + 1} \cdot j_{i+1}^{\ell m} (\Xi_\beta) \right] + \mathcal{V}_{A_{i}}^{\ell m} \cdot N_{A_{i}} \cdot \mathcal{V}_{j_{i+1}}^{\ell m} (\Xi_\beta)$$

$$= C_3 \cdot \left[ \mathcal{V}_{j_{i-1}}^{\ell m} \cdot N_{A_{i-1}} \left[ \frac{j_{i}^{\ell m} (\Xi_\beta)}{\Xi_\beta} \right] + \mathcal{V}_{A_{i}}^{\ell m} \cdot N_{A_{i}} \cdot \mathcal{V}_{j_{i+1}}^{\ell m} (\Xi_\beta) \right],$$

(2.10c)

where $j'(\xi) = \partial j(\xi)/\partial \xi$; $C_1$, $C_2$, $C_3$ are given by (A.2) and $\epsilon_{ijk}$ is Levi–Civita antisymmetric tensor (see eq. A.4 in H1). Above (see also eqs A.5 and A.8 in H1)

$$N_{A_{i}} = n_{a_1} \cdot n_{a_2} \cdots n_{a_l}.$$ (2.11)

The vectors $L_{im}^{\ell m}$, $M_{im}^{\ell m}$, and $N_{im}^{\ell m}$ in the STFT form are obtained from (2.10) by replacing $j_{i}^{\ell m} (\Xi)$ by $h_{i}^{\ell m} (\Xi)$ and $\mathcal{V}_{j_{i}}^{\ell m}$ by $\mathcal{V}_{A_{i}}^{\ell m}$.

3 Sources of elastic waves

Four different cases need to be considered which depend on the relations between reduced wavelength ($\Lambda$) and the size of a source zone ($R_0$) as well as the distance between the source and the receiver ($r$). These cases are $r > \Lambda$ and $r < \Lambda$ which we call far-field and near-field respectively, as well as $R_0 < \Lambda$ and $R_0 > \Lambda$, called a small source and a large source, respectively. If $R_0 \sim \Lambda$, the value of angular frequency approximately corresponds to the corner frequency for standard models of earthquake fault (Brune 1970; Evernden et al. 1986, p. 160). Thus, we obtain four possible combinations: (1) $r > \Lambda$, $R_0 < \Lambda$; (2) $r < \Lambda$, $R_0 < \Lambda$; (3) $r > \Lambda$, $R_0 > \Lambda$; and (4) $\Lambda < r < R_0$, $R_0 \sim \Lambda$ (see also Aki & Richards 1980, p. 804).

3.1 Far-field Radiation, Small Sources

Formulas for the far-field radiation are easy to obtain from general formulations in Appendix B by using only the first terms in curly brackets of (B.6), (B.8), (B.10), and (B.11). For instance, for the STF version of the displacement of order $l$ in the frequency domain, we obtain

$$q_{j}^{(l)}(\omega) = \frac{(-i\omega)^{l-1} \exp (i\omega/\alpha)}{4\pi(\lambda + 2\mu)\alpha^{l-1}(l-1)!} \cdot n_{A_{i}} N_{A_{i}} \mathcal{F}_{A_{i}}^{(l)}(\omega).$$

(3.1a)
where $\mathcal{F}^{(l)}$ is defined by (A.4); $\mathcal{P}^{(l+2)}$ is given by (A.13); $p(\omega)$ is the longitudinal spheroidal displacement function; $s(\omega)$ is the shear spheroidal displacement function; and $u(\omega)$ is the shear toroidal displacement function. The left superscript on $p$, $s$, and $u$ denotes the class $\eta$ of a source (2.6). These formulas, as well as time domain expressions which are easily obtainable (see Appendix B), can be used for calculating far-field body wave displacements. This solution can then be propagated using appropriate methods of the ray theory (Randall 1968; Bache & Harkrider 1976; Aki & Richards 1980; Barker & Minster 1980).

We summarize dynamic sources of elastic waves in Fig. 1 in a format similar to that of fig. 1 in H1. Here we do not show the sources connected specifically to the stress glut tensor moment; these are null sources both for static and dynamic displacements (Backus & Mulcahy 1976; Backus 1977). We also exclude from the diagram the non-internal sources.

The STF tensors for the sources of classes $\eta = 0$ and $\eta = 1$ (3.1a, b and e) are the same as
for the static excitation function (see H1, eqs 2.5 and 2.7). The combination of the STFT's for the sources of $\eta = 2$ class (A.13) is different from that of the static elastic source for $\ell > 1$ (H1, eq. 2.8). The tensor $\mathcal{F}_{\ell+2}^{(1)}$ (A.13) differs from the tensor $\mathcal{F}_{\ell-1}^{(1)}$ of H1 (eq. 3.20a) by a multiplier dependent on $\ell$. Thus, in the ‘small-source, far-field’ approximation, the dynamic elastic source of class $\eta = 2$ is the same as the source term of the Helmholtz or the Laplace equations (see H1, eq. 3.21). As we indicated in H1, only one STF moment tensor, $\mathcal{F}_{\ell=0}^{(2)}$ is non-zero for compression--dilatation centres (eq. 2.9 in H1). Thus the static and dynamic sources are equivalent for $\ell = 0$ (up to a scalar multiplier). The dissimilarity of the sources for $\ell > 1$ gives us, at least in principle, an opportunity to determine the equivalent-force tensors $\mathcal{F}_{\ell+2}^{(1)}$ and $\mathcal{F}_{\ell-1}^{(1)}$ if both the far-field radiation and the static deformation, or the near-field radiation (see Section 3.2 below) are known.

Average displacements can be easily computed (see section 3.4 in H1). For example,

$$r \langle 0_s^{(1)}(\omega) \rangle = \left[ \frac{r^2}{4\pi} \int \frac{\tilde{u}_0(t) \tilde{u}_0(t) d\omega}{12} \right]^{1/2}.$$  (3.2)

Then

$$r \langle 0_s^{(1)}(\omega) \rangle = \frac{1}{4\pi \mu} \cdot \left( \frac{\omega}{\beta} \right)^{l-1} \frac{l+1}{(l-1)! (2l+1)!!} \cdot |\mathcal{F}_{\ell}^{(1)}(\omega)|.$$  (3.3)

Here $|\mathcal{F}_{\ell}^{(1)}|$ is the scalar norm of the tensor $\mathcal{F}$ (see eq. A.11 in H1). The formula for $p$-wave has $l$ in place of $l + 1$ in the square bracket term in (3.3), $\alpha$ replaces $\beta$, and $(\lambda + 2 \mu)$ goes in place of $\mu$. For the double-couple source with the total scalar seismic-moment $M_0$ which increases as the Heaviside step-function with time, we obtain

$$\langle 0_s^{(2)}(\omega) \rangle = \frac{M_0 \sqrt{2}}{4\pi \mu \sqrt{5}}, \quad \text{and} \quad 0_s^{(2)} \max = \frac{M_0}{4\pi \mu \sqrt{3}},$$  (3.4)

since $|\mathcal{F}_{\ell}^{(1)}(\omega)| = \omega^{-1} M_0 \sqrt{2}$. Then $\langle 0_s^{(2)}(\omega) \rangle = \sqrt{0.4} \cdot 0_s^{(2)} \max$ (cf. Brune 1970, p. 5004; Aki & Richards 1980, prob. 4.6).

If we know the displacement, either from theoretical computations or from experimental measurements, we can obtain the multipole moments or the STF moment tensors by using the orthonormality of the vector spherical harmonics (eq. B.3 in H1). Thus, for example,

$$\delta^{(2)}_{\ell} \mathcal{F}_{\ell+1}^{(1)}(r - \alpha) = -4\pi \sqrt{5} \sqrt{1 - \frac{1}{4\pi \mu \sqrt{3}}} \int u_p(t - \alpha) Y_{p}^{\ell+1}(r) d\Omega,$$  (3.5)

(cf. Thorne 1980, p. 317) or

$$\delta^{(2)}_{\ell} \mathcal{F}_{\ell+1}^{(1)}(r - \alpha) = 4\pi \sqrt{3} \sqrt{1 - \frac{1}{4\pi \mu \sqrt{5}}} \int u_p(t - \alpha) Y_{p}^{\ell+1}(r) d\Omega,$$  (3.6)

where $u$ is the observed toroidal displacement and $f d\Omega$ is an integral over unit sphere.

Similarly, if we know that the source is of the $\eta = 0$ class, and of the order $l$

$$\delta^{(2)}_{\ell-1} \mathcal{F}_{\ell-1}^{(1)}(r - \alpha) = \frac{4\pi (\lambda + 2 \mu) \alpha^{l-1}!}{C_l (2l + 1)} \int 0_q^{(l)}(t - \alpha) Y_{q}^{R,lm}(r) d\Omega,$$  (3.7a)

and

$$\delta^{(2)}_{\ell-1} \mathcal{F}_{\ell-1}^{(1)}(r - \alpha) = \frac{r}{4\pi \mu \sqrt{5}} \cdot \frac{4\pi (\lambda + 2 \mu) \alpha^{l-1} (2l + 1)!!}{C_l (2l + 1)} \int 0_q^{(l)}(t - \alpha) Y_{q}^{R,lm}(r) d\Omega.$$  (3.7b)
In this paper we consider the measurements of seismic waves to be error-free. In this case we do not even need to invert the spheroidal $s$ waves, because the result will be identical to the one obtained from (3.7). The inversion formulas for transverse waves are analogous to (3.7).

Generally, spheroidal longitudinal or transverse waves of the $l$th order may be a mixture of waves of two even classes $\eta = 0$ and $\eta = 2$. Considering, for example, a multipole case in the frequency domain, there are two equations for $P$- and $S$-waves. Solving these equations we obtain

$$F^{l-1, lm}(\omega) = \frac{C_1(\lambda + 2\mu)P^{lm} + C_2\mu S^{lm}}{(-i\omega)^{l-1}[\lambda + \mu(3l + 1)]},$$

(3.8a)

and

$$F^{l+1, lm}(\omega) = \frac{(2l + 3)\mu(\lambda + 2\mu) - C_1S^{lm} - C_2P^{lm}}{(-i\omega)^{l+1}[\lambda + \mu(3l + 1)]},$$

(3.8b)

where

$$P^{lm}(\omega) = 4\pi r(\lambda + 2\mu)\alpha^{l-1}!! \int [p_{q}(\omega) + \gamma p_{q}(\omega)] Y^{R, lm}_{q} d\Omega,$$

(3.9a)

and

$$S^{lm}(\omega) = 4\pi r^2 \mu^{l+1}!! \int [s_{q}(\omega) + \gamma s_{q}(\omega)] Y^{S, lm}_{q} d\Omega.$$  

(3.9b)

Thus, we infer that for $l > 1$ it is possible, at least in principle, to obtain three source functions for each order $l$ on the basis of far-field radiation.

### 3.2 Near-field Radiation, Small Sources

The calculations in this case are similar to those for the long wavelength approximation considered earlier (Eqs 2.9). To avoid complicated equations we analyse only the toroidal displacement function for $r \to 0$.

$$u^{(l)}(\omega) = \frac{i\omega}{\beta \mu} \sum_{m=-1}^{m=1} M_{k}^{lm} \int_{x} \int_{T} f(x, t') \exp(i\omega t') \, dx \, dt',$$

(3.10)

(cf. B.5). For $M_{k}^{lm}$ we use the approximation (2.8b) instead of (B.1e). Then

$$\bar{u}_{f}^{(l)}(\omega) = \frac{l \cdot (2l - 1)!!}{4\pi (l + 1)! \mu^{l+1}!!} \cdot N_{A_{l-1}} N_{p} \cdot p_{q} \cdot F^{(l+1)}_{q A_{l-1}}(\omega),$$

(3.11)

which is the same as the static displacement function (eq. 2.7 in H1). However, the STF moment tensor $F$ is a function of the frequency (Petrashen' 1978, eq. 15.112).

Since longitudinal and transverse waves arrive at nearly the same time and almost cancel at near range, we use their appropriate combinations (see Eqs 2.9 and the discussion below it) which correspond to $0u^{(l)}$ and $\bar{u}_{f}^{(l)}$ of static displacement (H1). The results for functions $0u^{(l)}(\omega)$ and $\bar{u}_{f}^{(l)}(\omega)$ are similar to those discussed above. The displacement depends on the distance, $r$, as $r^{-l-\eta}$; for the first- and second-rank sources, this dependence has been pointed out by Wu & Ben-Menahem (1985).
3.3 Far-field radiation, large sources

In the expansion (3.1) we use only the first term for the point-like sources in Appendix B (see eq. B.1). To obtain the displacement functions for sources comparable in size with a wavelength, we keep additional terms in the expansions. Thus, for example,

$$T_{l-1,lm}^{f}(\omega) = D^{(l)} \cdot \int_{X} f_{p}(x, \omega) Y_{l-1,lm}^{*} j_{l-1}(\xi) \, dx$$

$$= D^{(l)} \cdot \int_{X} R_{l-1}^{f} f_{p}(x, \omega) Y_{l-1,lm}^{*} \left[ 1 + \sum_{k=1}^{\infty} U_{lk}^{(k)} \left( \frac{R \omega}{v} \right)^{2k} \right] \, dx, \quad (3.12)$$

here $U_{lk}$ is defined by (B.2), and $D^{(l)}$ is given by (A.5). The first term of (3.12), which is independent of the wave velocity $v (v = \alpha \beta)$, corresponds to the multipole moment $F_{l-1,lm}$ (see Eq. A.3). We obtain expressions, for instance, for the p-wave displacement of this source ($\eta = 0$) by inserting $T_{p}^{l-1,lm}$ in place of $F_{l-1,lm}$ in the formula (B.8). We obtain two other multipoles similar to (3.12). These sources have $2l + 1$ degrees of freedom each.

From (3.12), the multipole $T_{l-1,lm}^{f}$ (see Eq. 2.5) is the Hankel transform (Morse & Feshbach 1953) of the projection of equivalent-force $f$ on the vector spherical harmonic $Y_{l,lm}$ (Bosco & Sacchi 1981). The new multipole moment depends on the value of wave velocity in a medium, so we need separate moments to calculate spheroidal compressional and transverse displacements. Thus the multipole expansion is less useful. From (3.12) we also see that, since $\alpha > \beta$, for sources of finite size the P-wave spectrum is richer in high frequencies than the S-wave spectrum (cf. Aki & Richards 1980, pp. 820–825; Hanks 1981; Silver 1983).

For purposes of future discussion we denote

$$kF_{l,lm}^{f}(\omega) = D^{(l)} \cdot \int_{X} R_{l}^{f} f_{p}(x, \omega) Y_{l,lm}^{*} X^{lk} \left( \frac{R \omega}{v} \right)^{2k} \, dx, \quad (3.13)$$

as the $k$th-level multipole moment, then $0F_{l,lm}^{f} = F_{l,lm}^{f}$ (see Appendix A). In (3.13) $X^{lk}$ is either $U_{lk}$, or $V_{lk}$, or $W_{lk}$ from (B.2).

Converting (3.12) into the STF notation we obtain

$$T_{l-1,lm}^{f} = D^{(l)} \cdot C_{1} \cdot \sum_{A_{l}} \mathcal{A}_{A_{l}}^{lm} \left\{ \mathcal{A}_{A_{l}}^{f(1)} + \sum_{k=1}^{\infty} U_{lk}^{(k)} \Lambda_{A_{l}}^{2k} \cdot 2 \mathcal{A}_{A_{l}}^{(1+2k)} \right\}, \quad (3.14)$$

where $\Lambda_{A_{l}}$ is given by (2.3); the sources $2 \mathcal{A}_{A_{l}}^{(1+2k)}$ are defined in H1 (eq. 3.12b). For the $k = 1$ source see also (A.12). Using, for example, the STFT expansion of the seismic-moment tensors for the circular fault model of H1 (eq. 4.43), we calculate amplitudes of far-field shear waves caused by five non-zero sources. [The last term in eqs. 4.43 of H1 should be read as $\mathcal{A}_{A_{l}}^{(2)} = A/R_{0}^{2}$. In counting sources we do not consider $2 \mathcal{A}_{A_{l}}^{(4)}$ as a separate source (see 3.14).] The sum of these oscillations is equal to that obtained by Silver (1983, p. 1504 – see also Doornbos 1982) for a planar circular earthquake fault.

In contrast to static sources, dynamic sources consist of an infinite number of STFTs (for example, $2 \mathcal{A}_{A_{l}}^{(1+2k)}$ for the level $k = 0, 1, 2, \ldots$, in Eq. 3.14). Thus, we need to distinguish between the classes of sources of elastic deformation (there are three such classes) and the classes of the STFTs (or source terms), which in principle, are infinite in number. Since all the STFTs are part of the linear combination (3.14), which itself is part of the excitation...
formula (B.11) for elastic waves, it is impossible to distinguish between these moment tensors in the absence of additional constraints. We might describe the elastic radiation in terms of the first non-zero term in the source function as 'shielding' or 'shadowing' other terms in this expansion. As in Eqs (3.12) and (3.14), we use other formulas from Appendix B to obtain displacement expressions for large sources of other classes.

From (3.14) and (B.1) we infer that, unlike the sources of static deformation (see H1, section 3.3), there are no null classes of the STFT terms for sources of elastic waves. All equivalent-force moment tensor sources are active. The dynamic equivalent-force moment tensor terms of all classes 'radiate' seismic waves: their amplitude diminishes as $X^{2k} \cdot (R_0/A_p)^{2k}$. As we mentioned earlier, $(R_0/A_p) \approx 1$ in the vicinity of the corner frequency. Thus, if we ignore the contributions from the first- $(k = 1)$ and higher-level terms in (B.1) at this frequency, we introduce a fractional error of the size $X^{2k} = 1/[2(l + 1)]$. The seismic-moment tensors $\mathcal{R}^{(l+2k)}$ (see H1, section 3), however, have terms which are null sources, i.e. it is possible to form one or two linear combinations of these terms which produce no displacement outside a source region (H1; Backus & Mulcahy 1976; Backus 1977).

In Fig. 1, the dynamic sources of the class $\eta = 0$ include all the left-hand points of the same order, taken from representations of even classes, beginning with the zeroth class up to infinity. As we mentioned earlier, only equivalent-force tensors are included in the source; they are connected by solid lines in the diagram. The source terms of the class $\eta = 2$ correspond to all the right-hand points in the diagram. For 'spheroidal' tensors of class two and higher, each tensor class contains representatives of two classes of sources, i.e. $\eta = 0$ and $\eta = 2$. These representatives are proportional either to $k^{F^{-1},lm}$ or to $k^{F^{+1},lm}$ (3.13) or to their respective STFT equivalents.

In the dynamic far-field case the tensor $2\mathcal{R}^{(l+2)}$ causes displacement of the class $\eta = 0$. For the static case (see Eq. B.12), as well as the near-field dynamic case (see previous Section), this source excites the static displacement $u^{(l)}$ (see eq. 2.8 in H1), so it belongs to a source of the class $\eta = 2$.

We summarize the transformation of the sources of elastic waves when we analyse far-field vs near-field (or static) observations as follows. In the infinite series of the STFTs for classes $\eta = 1$ and $\eta = 2$, all their terms with exception of the first (zeroth-level) become null sources. The source of the $\eta = 0$ class loses all but the first two tensors; its second (first-level) term then joins the first tensor of the $\eta = 2$ class to form the second static spheroidal source. As we mentioned earlier, the above transformation allows us, in principle, to gain additional information by comparing the $\eta = 2$ sources of far-field vs near-field (or static) deformation. It is interesting that for the Laplace equation all the terms $k^{F_{l-1},lm} (k > 1)$ are null sources (see discussion around eq. 3.22 in H1). Thus no new information can be obtained by comparing the sources of the Laplace (static) and the Helmholtz (dynamic) equations.

3.4 NEAR-FIELD RADIATION, LARGE SOURCES

It is relatively easy to obtain relevant formulas for the near-field radiation of relatively large sources, i.e. $R_0 < \Lambda$. We should repeat considerations of Section 3.3, but the source should contain several multipole components. Calculations are the same as in the static case (H1). If the size of the source region is comparable or larger than the reduced wavelength, the approximations (2.8) are no longer convenient to use, so one has to apply the original form of the Hansen vectors (2.10).
4 Seismic sources of zeroth to second order

In this section we review a few low order sources. These sources are usually special cases, given the indigenous nature of the source or other constraints. As a result, the zeroth-level tensor moment representative of sources (corresponding to \( k = 0 \) in 3.13) are not available. This absence makes room for ‘unshielded’ sources which can be inverted from a seismogram. In this section we assume that the seismic radiation is recorded in an intermediate- or far-field range.

Formulas (B.10) and (B.11) show that the amplitude spectrum of the far-field displacement for sources which belong to the seismic-moment tensor of the rank \( L \) is multiplied by a factor \( \omega^{-L} \). If we consider only those sources which have a factor of \( \omega^2 \) or less, when compared with the second-rank tensor radiation, we have 10 possible internal sources to analyse (see Fig. 1). These sources correspond to the seismic-moment tensor up to the fourth-rank. In the static case (H1), the displacement caused by sources of the rank \( L \) depends on distance as \( r^{-L} \). Therefore, we have primarily considered and compared tensor representations which belong to a seismic-moment tensor of the same rank, which corresponds to the vertical alignment of points in Fig. 1. For the dynamic case, in a far-field approximation the distance dependence is the same for all tensor representations (1/r). Thus it is more advantageous to discuss the representations of the same weight or order \( l \), i.e., to move horizontally in Fig. 1.

4.1 MONOPOLE SOURCES

The case of compression–dilatation centres (isotropic sources) is a special (degenerate) case of (B.11c). Only compressional waves are excited for monopole source, so the moment tensor has the form

\[
(2) = \mu \mathbf{F}(2).
\]

See (A.13) for notation. Then the displacement is

\[
2p_j^{(0)}(t) = -\frac{1}{12\pi r(l + 2\mu)\alpha} n_i \left[ \partial_t \mathbf{F}(2)(t - r/\alpha) + \frac{\alpha}{r} \mathbf{F}(2)(t - r/\alpha) \right],
\]

which corresponds to solutions by Petrashen' (1978, eqs 12.20, 12.4) and by Ben-Menahem & Singh (1981, eq. 4.208).

A source composed of shear dislocations presents a more interesting case. We shall call such sources deviatoric sources. The scalar \( \mu \mathbf{F}(2) \) is identically equal to zero for such a source; the first non-zero monopole source is that of \( \mu \mathbf{F}(4) = \mathbf{F}(4) \) (see eqs 4.39–4.41 in H1). For such a source we obtain

\[
2p_j^{(0)}(t) = -\frac{1}{120\pi r(l + 2\mu)\alpha^2} n_i \left[ \partial_t \mathbf{F}(4)(t - r/\alpha) + \frac{\alpha}{r} \partial_t \mathbf{F}(4)(t - r/\alpha) \right].
\]

Thus, complex deviatoric earthquake sources may still cause the monopole radiation. But in the frequency domain, the source’s spectrum should rise as \( \omega^2 \) compared to the radiation caused by the deviatoric part of the standard seismic-moment tensor. A picture of equivalent forces for this source was described in H1 (see text below eq. 4.41). The source does not produce any static displacement outside a source zone (H1). In our next contribution (Kagan 1987b), we shall estimate the strength of a \( \mathbf{F}(4) \) source for several deterministic and stochastic models of earthquake rupture. In particular, we will show that this source is
zero for most standard models of earthquake faults. Only *en echelon* fault pattern yields a non-zero contribution for this source.

### 4.2 Dipole Sources

Two of the simplest dipolar sources, a single-force source and a rotation centre, are not internal sources (see fig. 1 in H1). Thus, they are not considered here. Internal sources of dipole radiation are hence $F^{2,1m}$, $F^{0,1m}$, and $F^{1,1m}$ (see Eq. 3.13 and Fig. 1). Displacement functions for these sources are as follows (see Appendix B):

\[
op_j^{(1)}(\omega) = \frac{(-i\omega)^2 \exp(i\omega/\alpha)}{8\sqrt{3}\pi r(\lambda + 2\mu)\alpha^2} \sum_{m=-1}^{m=1} 1^{F^{0,1m}} \left[ Y_{j,m}^{R,1m} \frac{\sqrt{3}i\alpha}{r\omega} \cdot Y_{j,m}^{2,1m} \left( 1 + \frac{i\alpha}{r\omega} \right) \right],
\]

\[
o_j^{(1)}(\omega) = \frac{(-i\omega)^2 \exp(i\omega/\beta)}{4\sqrt{6}\pi r\mu^2} \sum_{m=-1}^{m=1} 1^{F^{0,1m}} \left[ Y_{j,m}^{S,1m} \frac{\sqrt{3}i\beta}{r\omega} \cdot Y_{j,m}^{2,1m} \left( 1 + \frac{i\beta}{r\omega} \right) \right],
\]

\[
2p_j^{(1)}(\omega) = \frac{(-i\omega)^2 \exp(i\omega/\alpha)}{10\sqrt{6}\pi r(\lambda + 2\mu)\alpha^2} \sum_{m=-1}^{m=1} 1^{F^{2,1m}} \left[ Y_{j,m}^{R,1m} + \frac{\sqrt{3}i\alpha}{r\omega} \cdot Y_{j,m}^{2,1m} \left( 1 + \frac{i\alpha}{r\omega} \right) \right],
\]

\[
2\phi_j^{(1)}(\omega) = \frac{(-i\omega)^2 \exp(i\omega/\beta)}{20\sqrt{3}\pi r\mu^2} \sum_{m=-1}^{m=1} 1^{F^{2,1m}} \left[ Y_{j,m}^{S,1m} - \frac{\sqrt{3}i\beta}{r\omega} \cdot Y_{j,m}^{2,1m} \left( 1 + \frac{i\beta}{r\omega} \right) \right],
\]

\[
l_j^{(1)}(\omega) = \frac{(-i\omega)^3 \exp(i\omega/\beta)}{40\pi r\mu^2} \sum_{m=-1}^{m=1} 1^{F^{1,1m}} Y_{j,m}^{1,1m} \left( 1 + \frac{i\beta}{r\omega} \right).
\]

In the static limit (H1) both the spheroidal sources $F^{2,1m}$ and $F^{0,1m}$ are equivalent sources (the deformation caused by them is identical, or belongs to the same $\eta = 2$ class and to the same first order, $l = 1$). Thus we can form two linear combinations of these sources such that one combination is a null source (cf. eq. 3.22 in H1). The equivalent-force distribution for these sources is described in section 4.2.1 of H1. From (4.4) we see that *dynamic* sources $F^{2,1m}$ and $F^{0,1m}$ produce displacements of two different spheroidal classes. The radiation pattern of the latter source is that of a single-force ($Y^{0,1m}$), but the frequency dependence for the source $F^{0,1m}$ has a factor $\omega^2$ compared to the single-force solution. The radiation pattern of the source $F^{2,1m}$ is that of $Y^{2,1m}$ harmonic (see eq. B.1 in H1).

From (4.4) we infer that if the sources $F^{0,1m}$ and $F^{2,1m}$ have an equal strength, their radiation should be of comparable amplitude. Using a procedure similar to that described in Section 3.1 (see Eqs 3.9), we can invert spheroidal compressional and transverse waves and then obtain both dipole sources $F^{0,1m}$ and $F^{2,1m}$ from seismic waves. As we pointed out in section 4.2.2 of H1 (see also Backus 1977), if we impose a deviatoric condition on the second-rank stress glut density in the source, the number of spheroidal dipole sources is reduced to one. The sources $F^{2,1m}$ and $F^{0,1m}$ are linearly related in this case. A linear combination of the dipolar sources $F^{2,1m}$ and $F^{0,1m}$, representations of the fifth-rank seismic-moment tensor (see Fig. 1), is not shielded in this case. It may be inverted, at least in principle, from seismic data. A possible earthquake fault pattern responsible for the dipolar radiation is shown in fig. 3b of H1. The sources $F^{2,1m}$ and $F^{0,1m}$ may be distinct.
only for explosive sources in the presence of random stress heterogeneities (cf. Stevens 1980). The difference between them may have value as a discrimination factor.

The toroidal source \( F^{1,1m} \) (cf. eq. 4.38 in H1) can also be inverted from (4.4e). Since the zeroth-level representative of the source, a rotation centre \( F^{1,1m} \), is not indigenous, \( F^{1,1m} \) is the lowest-level dynamic toroidal source. The displacement spectrum of this source has a factor proportional to \( \omega^2 \) when compared with the radiation of a second-rank source. The simplest picture of equivalent forces for this source is presented in H1 (see discussion below eq. 4.41).

4.3 QUADRUPOLE SOURCES

The quadrupole sources (see Fig. 1) are the first low-order internal sources which exhibit standard behaviour, i.e. the zeroth-level representatives (corresponding to \( k = 0 \) in 3.13) of these sources are indigenous and are present in all three source classes (see Fig. 1). We can use formulas from Section 3 to calculate excitation functions for all these sources and use Eqs 3.8 and 3.9 to obtain source parameters through an inversion. As we explained in H1, the quadrupole sources are also the first sources having an internal structure: one degree of freedom out of five, which can be used as a structure parameter of a deviatoric tensor. Sources of higher order have a larger number of parameters.

The spectrum of displacement of the \( F^{1,2m} \) source is flat near \( \omega = 0 \) for the standard model of earthquake fault rupture, i.e. a scalar seismic-moment increasing as the step-function of time. (For the step-function the spectrum is to decrease as \( \omega^{-1} \). Combined with the \( \omega \) increase of the radiation spectrum for the second-rank source (3.1), it yields the flat spectrum near \( \omega = 0 \).) The toroidal source \( F^{2,2m} \) should exhibit a \( \omega \) raise in the neighbourhood of zero frequency. For the second spheroidal source \( F^{3,2m} \), the spectrum should be proportional to \( \omega^2 \) near zero frequency (cf. Doornbos 1982; Silver 1983). These predictions are confirmed, for example, by theoretical calculations of the spectra of quadrupole radiation of an explosive source in a non-uniform stress field (Stevens 1980, p. 318). The numbering of the radiation sources is inverted compared to our results: Stevens' first source corresponds to our \( F^{3,2m} \) source, his second spheroidal source is our \( F^{1,2m} \).

In summary, the second-rank seismic-moment tensor has six degrees of freedom (see Fig. 1), the third-rank tensor has 18 degrees of freedom, the fourth-rank tensor has 24 degrees of freedom. The appropriate numbers of degrees of freedom for a deviatoric source are 5, 15, and 25 respectively. For the fourth-rank tensor the number increases since the monopole source is transferred to it from the second-rank tensor (see Section 4.1).

To check whether our formulas are correct and complete, in addition to the tests described in Appendix B, we also repeated the simulation tests similar to those reported at the end of section 4 of H1. We compared far-field seismic waves excited by several randomly distributed second-rank sources with the far-field waves calculated using the decomposition of the sources into multipoles of second-, third-, and fourth-ranks (see the paragraph above). The time dependence of the second-rank elementary sources is taken to be the step-function. The difference between these two computations decays as \( \omega^3 \) for frequency values approaching zero.

5 Discussion

One of the major difficulties in inverting the higher-rank seismic-moments has been the high number of degrees of freedom needed to describe such moments (Madariaga 1983). However, as shown by the present study, some sources which are part of these higher-rank
moments have a relatively small number of intrinsic parameters. For example, the two dipolar sources $F^{2,1m}$ and $F^{0,1m}$ described in Section 4.2 have only six degrees of freedom. The spectra of the waves excited by those sources differ by a factor proportional to $\omega$ from the spectrum of radiation of second-rank sources. Their dipolar radiation pattern makes them relatively easy to invert, since relatively few stations are needed for a reasonably complete coverage. As we mentioned in Section 4.2, for the deviatoric sources, there is only one dipolar source which has $\omega$-dependence. Thus the total number of degrees of freedom for this source is three. Actually, in some instances the strength of the dipole sources is so high that either it becomes a dominant source or even the first arrivals of $P$-waves exhibit a dipolar pattern (see Knopoff & Gilbert 1960).

In H1 we have already discussed the scale-invariant property of the Navier equations. This property makes it impossible to infer any scale factor from the inversion of seismic wave data to sources of these waves (cf. Bosco & Sacchi 1981; Biedenharn & Louck 1981, p. 439). To obtain spatial or temporal parameters of the sources, we need additional hypotheses on the nature of the source. These assumptions may take the form of specific models of earthquake rupture (see, for example, Aki & Richards 1980; Doornbos 1982; Stump & Johnson 1982). Or they may have a more general form, for instance, concentration of the seismic sources in certain time-space points, intervals, or other subsets. These models might work well if a seismic record consists of clearly separated wave arrivals from different seismic events; when such records interfere, their interpretation becomes more ambiguous (cf. Kostrov 1975). Another frequently made assumption is that the normalized time history (or the normalized spectrum) of all multipole displacements is identical. Our investigations (Kagan & Knopoff 1985) seem to indicate that the complexity of earthquake rupture increases towards the end of the process. Since higher order multipoles reflect the complexity of a source time-space function, it is probable that time maxima of these multipoles are shifted towards the end of the fracture.

Even with these assumptions, the inversion may be unstable. Kostrov (1975, Ch. III.7) shows, for instance, that if we assume the rupture to be on a planar surface, it is possible to obtain a complete time–space history for it by antennae synthesis. Unfortunately, the inversion is unstable with regard to small errors. The rupture process can be restricted to an arbitrarily small path of the plane but even so, seismograms differ only negligibly from the initial full-size consideration (see also Madariaga 1983, p. 41; Aki, Chen & Zeng 1986, and references therein).

In addition to spatial seismic-moment tensors which we have studied here and in H1, Backus & Mulcahy (1976, pp. 357–358) and Backus (1977) introduced temporal seismic-moment tensor moments. These temporal moments can be easily determined if one finds multipoles or STFTs by inverting seismic records (see, for example, the end of Section 3.1). Unfortunately, in the most interesting cases a source zone cannot be considered point-like; the inversion of the second-rank seismic-moment tensor would yield a sum of the second temporal moment of the tensor and of higher-level quadrupole moment tensors as described by (3.12–3.14) (cf. Doornbos 1982; Silver 1983). Amplitudes of seismic waves excited by the sources corresponding to these moments are roughly of the same size (Backus 1977, p. 6). However, as has been explained above, the sources corresponding to the higher-level spatial tensors are shielded by the representation corresponding to $k=0$ in Eq. 3.12 (the solid circle marked by $F^{1,2m}$ in Fig. 1). Thus, separating the spatial and temporal moments is impossible without additional assumptions about the nature of the source zone. We discuss this problem in more detail in our companion paper (Kagan 1987b).

We shall briefly comment on the method of determining the size of an earthquake focal zone based on corner frequency $\omega_{cr}$ (see, for example, Hanks 1981; Silver 1983; Andrews
Estimating focal size is based on the standard models of earthquake fault such as shear failure on a circular fault (Brune 1970; Evernden et al. 1986, p. 160). As we mentioned above, no measurements of spatial scale of a focal region can be made without some assumptions. For standard models of earthquake rupture (involving step- or ramp-function of the scalar seismic-moment release), the spectra of higher-rank seismic-moment tensors depend on the frequency as $\omega^{L-2}$. The third-rank sources have radiation proportional to $\omega$ near zero frequency and therefore should have peaked spectra. From Eqs (3.1) it is clear that the peaks of spectra of higher-rank sources should be even higher (cf. Stevens 1980). As we mentioned earlier, complex earthquakes may have relatively strong higher-rank sources. If seismic waves are recorded by one station or several stations, the corner frequency may be far displaced due to the influence of the higher-rank seismic sources. Similarly, the high-frequency decay of the spectrum may be influenced by the higher-rank sources. This variability of the high-frequency part of the spectrum of earthquakes may present an additional difficulty in discriminating complex earthquakes from explosive sources (Evernden et al. 1986).

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**References**


Appendix A: the STF source tensors

As we mentioned in H1, we use two kinds of vector spherical harmonics: pure-orbital harmonics (Thorne 1980) which describe the source, and pure-spin harmonics which are more suitable for describing wave radiation or normal-mode displacements. In the dynamic case we make greater use of pure-spin harmonics. These harmonics can be written in the STFT form (Thorne 1980)

\[ Y_{j}^{R,lm} = h_{j}^{l} R_{j}^{lm} A_{l+1}^{*} A_{l}^{*} \] (A.1a)
\[ Y_{j}^{S,lm} = C_{3} \cdot [ \mathcal{R}_{j}^{lm} A_{l+1}^{*} N_{l+1}^{*} - h_{j}^{l} A_{l-1}^{*} N_{l-1}^{*} ], \] (A.1b)
\[ Y_{j}^{T,lm} = C_{3} \cdot e_{j p q} \eta_{p} \mathcal{V}_{q A_{l-1}^{*} N_{l-1}^{*} A_{l-1}^{*} A_{l}^{*} } \] (A.1c)

where

\[ C_{1} = \sqrt{\frac{l}{2l+1}}, \quad C_{2} = \sqrt{\frac{l+1}{2l+1}}, \quad \text{and} \quad C_{3} = \sqrt{\frac{l}{l+1}}, \] (A.2)

(see Eqs 2.10 and 2.11, as well as eqs B.1 and B.4 in H1). Here \([ \cdot ]^{T}\) signifies a transverse part of the expression in the square brackets (Thorne 1980, p. 308).

The multipole moments are

\[ F_{l+1,lm}^{(l)} = D^{(l)} \cdot \int_{X} f_{p}(x) Y_{p}^{l+1,lm} R_{l+1}^{*} dx = D^{(l)} \cdot C_{1} \cdot \mathcal{Y}_{l}^{lm} \mathcal{F}_{l+1}^{(l)}, \] (A.3)

where the STF multipole moment tensor is (eq. 2.6 in H1)

\[ \mathcal{F}_{l}^{(l)} = \left[ \int_{X} f_{j}(x) X_{A_{l-1}^{*}} \right]^{STF} C_{1} \cdot \sum_{m=-l}^{m=l} \mathcal{Y}_{l}^{lm} X_{l-1,lm}, \] (A.4)

and

\[ D^{(l)} = \frac{4\pi l!}{(2l+1)!!}. \] (A.5)

The equivalent force \( f(x) \), multipole \( F_{l,lm}^{(l)} \) and the STFT \( \mathcal{F}_{l}^{(l)} \) in Eqs (A.3–A.15), are functions of time or frequency. The multipole and the STFT expansions in (A.3) and (A.4) are transformed as follows:

\[ \sum_{m=-l}^{m=l} Y_{l-1,lm}^{(l)} F_{l-1,lm}^{(l)} = \frac{l}{2l+1} \cdot \mathcal{F}_{l+1}^{(l)} A_{l-1}^{*} N_{l-1}^{*} \] (A.6)

Similarly, for toroidal sources we obtain

\[ F_{l,lm}^{(l+1)} = -D^{(l)} \cdot \int_{X} f_{p}(x) Y_{p}^{l,lm} R_{l+1}^{*} dx = i \cdot D^{(l)} \cdot C_{3} \cdot \mathcal{Y}_{l}^{lm} \mathcal{F}_{l+1}^{(l+1)}, \] (A.7)

and

\[ \mathcal{F}_{l+1}^{(l+1)} = \left[ \int_{X} f_{p}(x) X_{p A_{l-1}^{*}} e_{p q} \right]^{STF} C_{3} \cdot \sum_{m=-l}^{m=l} \mathcal{Y}_{l}^{lm} F_{l,lm}^{(l+1)}, \] (A.8)

Then, as in (A.6), two expansions are transformed as

\[ \sum_{m=-l}^{m=l} Y_{l,lm}^{(l)} F_{l,lm}^{(l+1)} = \frac{l}{l+1} \cdot \mathcal{F}_{l+1}^{(l+1)} N_{l+1}^{*} A_{l+1}^{*} e_{p q}. \] (A.9)
Finally, for $F^{l+1,lm}$ sources

$$F^{l+1,lm} = D^{(l)} \cdot \int_X f_p(x) Y_p^{l+1,lm} R_x \, dx = D^{(l)} \cdot C_2^{-1} Y_{A_1}^{lm} \left( \frac{l}{2l+1} \cdot 2 \mathcal{F}_{A_1}^{(l+2)} - \mathcal{F}_{A_1}^{(l+2)} \right),$$

where the STF multipole moment tensors are

$$\mathcal{F}_{A_1}^{(l+2)} = \left[ \int_X f_p(x) X_{pA_1} \, dx \right]^{STF},$$

and

$$2 \mathcal{F}_{jA_{l-1}}^{(l+2)} = \left[ \int_X f_j(x) X_{jA_{l-1}} \, dx \right]^{STF},$$

where the left superscript denotes a version number of a STF tensor (see also eq. A.10 in HI). If we denote

$$\mathcal{G}_{A_1}^{(l+2)} = \frac{l}{2l+1} \cdot 2 \mathcal{F}_{A_1}^{(l+2)} - \mathcal{F}_{A_1}^{(l+2)},$$

then

$$\mathcal{G}_{A_1}^{(l+2)} = C_2 \cdot \sum_{m=-l}^{m=l} Y_{A_1}^{lm} F^{l+1,lm},$$

and

$$\sum_{m=-l}^{m=l} Y_{A_1}^{l+1,lm} F^{l+1,lm} = \frac{l}{l+1} \cdot \mathcal{G}_{iA_{l-1}}^{(l+2)} N_{A_{l-1}} - \frac{2l+1}{l+1} \cdot \mathcal{G}_{A_1}^{(l+2)} N_{A_{l-1}}.$$

### Appendix B: point sources

In this appendix we derive formulas for point-like models of dynamic sources of seismic waves. For this purpose we must find formulas for the Hansen vectors which define the dynamic Green's tensor (2.2). There are four possible forms of these Hansen vectors: spectral vs temporal as well as vector harmonics vs the STFTs. Since too much space is required to show all these forms for all sources, we will only give expressions for one full set of source excitation functions. The transformation rules are simple enough to allow the easy derivation of the remaining sets of formulas.

The Hansen source vectors in spectral form are (see eq. 2.4)

$$L_{l+1,lm}^{l-1} = \frac{C_1}{(2l-1)!!} Y_{l-1,lm}^{l-1} \left( \frac{R \omega}{\alpha} \right)^{l-1} \left[ 1 + \sum_{k=1}^{\infty} U_{l+1}^{l-1,lm} \left( \frac{R \omega}{\alpha} \right)^{2k} \right],$$

$$L_{l+1,lm}^{l+1} = \frac{C_2}{(2l+3)!!} Y_{l+1,lm}^{l+1} \left( \frac{R \omega}{\alpha} \right)^{l+1} \left[ 1 + \sum_{k=1}^{\infty} U_{l+1}^{l+1,lm} \left( \frac{R \omega}{\alpha} \right)^{2k} \right],$$

$$N_{l+1,lm}^{l-1} = \frac{C_2}{(2l-1)!!} Y_{l-1,lm}^{l-1} \left( \frac{R \omega}{\beta} \right)^{l-1} \left[ 1 + \sum_{k=1}^{\infty} U_{l+1}^{l-1,lm} \left( \frac{R \omega}{\beta} \right)^{2k} \right].$$
Point sources of elastic deformation

\[ N_{s+1,lm} = -\frac{C_1}{(2l+3)!!} Y^{l+1,lm} \left( \frac{R \omega}{\beta} \right)^{l+1} \left[ 1 + \sum_{k=1}^{\infty} V^{lk} \left( \frac{R \omega}{\beta} \right)^{2k} \right], \]  

\[ M_{s}^{lm} = \frac{1}{(2l+1)!!} Y^{l,lm} \left( \frac{R \omega}{\beta} \right)^{l} \left[ 1 + \sum_{k=1}^{\infty} W^{lk} \left( \frac{R \omega}{\beta} \right)^{2k} \right]. \]  

where \( C_1, C_2 \) and \( C_3 \) are defined by (A.2), and

\[ U^{lk} = \frac{(-1)^{k}(2l-1)!!}{2^{k}k!(2l+2k-1)!!}, \quad V^{lk} = \frac{(-1)^{k}(2l+3)!!}{2^{k}k!(2l+2k+3)!!}, \]  

and

\[ W^{lk} = \frac{(-1)^{k}(2l+1)!!}{2^{k}k!(2l+2k+1)!!}. \]  

The outside Hansen vector terms are

\[ L_{lm}^{im} = \frac{\alpha(-i)^{l} \exp (i\omega_{r}/\alpha)}{\omega r} \left[ C_1 Y^{l+1,lm} \left[ 1 + \sum_{k=1}^{l+1} A^{lk} \left( \frac{i \alpha}{r \omega} \right)^{k} \right] \right], \]  

\[ N_{lm}^{lm} = \frac{\beta(-i)^{l} \exp (i\omega_{r}/\beta)}{\omega r} \left[ C_2 Y^{l+1,lm} \left[ 1 + \sum_{k=1}^{l+1} B^{lk} \left( \frac{i \beta}{r \omega} \right)^{k} \right] \right], \]  

\[ M_{lm}^{lm} = \frac{\beta(-i)^{l+1} \exp (i\omega_{r}/\beta)}{\omega r} Y^{l,lm} \left[ 1 + \sum_{k=1}^{l} C^{lk} \left( \frac{i \beta}{r \omega} \right)^{k} \right], \]  

where

\[ A^{lk} = \frac{(l+k-1)!}{2^{k}k!(l-k-1)!}, \quad B^{lk} = \frac{(l+k+1)!}{2^{k}k!(l-k+1)!}, \quad C^{lk} = \frac{(l+k)!}{2^{k}k!(l-k)!}. \]  

For vector displacement functions, which we call \( \mathbf{p} \) and \( \mathbf{s} \) (see also eqs 2.3–2.8 in H1), for the spheroidal longitudinal and the spheroidal transverse waves, respectively, we obtain, for example,

\[ \alpha^{(l)}(\omega) = \frac{i \omega}{\alpha(\lambda + 2\mu)} \sum_{m=-l}^{m=l} L_{lm}^{lm} \int_{X} \int_{T} L_{s+1,lm}^{l+1,lm} f(x, t') \exp (i \omega t') \, dx \, dt'. \]  

Here \( t' \) is time measured at a source region. We assume that the time variation of the source forces is \( \exp (i \omega t) \). From here on in this appendix, we will consider a source to be point-like. We only take the first (zeroth-level, see Eq. 3.13) term into account in decompositions (B.1).
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Then

\[0_p^{(l)}(\omega) = \frac{C_1(2l+1)}{4\pi\epsilon(\lambda + 2\mu)\alpha^{l-1}} \sum_{m=-l}^{l} F^{l-1,lm}(\omega) \cdot \left\{ Y^{l,lm} + \sum_{k=1}^{l-1} \left[ C_1 Y^{l-1,lm} A^{lk} - C_2 Y^{l+1,lm} B^{lk} \left( \frac{\alpha}{r\omega} \right)^k \right] \right\}

- C_2 Y^{l+1,lm} \left[ (2l+1)!! \left( \frac{\alpha}{r\omega} \right)^l \left( 1 + \frac{\alpha}{r\omega} \right) \right], \quad (B.6)\]

where \(F^{l-1,lm}(\omega)\) is given by (A.3). From (B.6) we obtain the displacement function in the time domain by using the relation (cf. Aki & Richards 1980, p. 129; Thorne 1980, p. 321)

\[\partial_t^l u(t) = \frac{1}{2\pi} \int u(\omega) (-i\omega)^l \exp(-i\omega \tau) d\omega. \quad (B.7)\]

Thus (Ben-Menahem & Singh 1981, eq. 4.26),

\[0_p^{(l)}(t) = \frac{C_1(2l+1)}{4\pi\epsilon(\lambda + 2\mu)\alpha^{l-1}} \cdot \sum_{m=-l}^{l} \left\{ Y^{l,lm} \delta_t^{l-1} F^{l-1,lm}(t - r/\alpha) \right\} \cdot \left[ \sum_{k=1}^{l-1} \left[ \partial_t^{l-k-1} F^{l-1,lm}(t - r/\alpha) C_1 Y^{l-1,lm} A^{lk} - C_2 Y^{l+1,lm} B^{lk} \left( \frac{\alpha}{r} \right)^k \right] \right]

- C_2 Y^{l+1,lm} \left[ (2l+1)!! \left( \frac{\alpha}{r} \right)^l \int_{r/\alpha}^{+\infty} \tau F^{l-1,lm}(t - \tau) d\tau \right]. \quad (B.8)\]

The limits in the integral of (B.8) should be understood as a symbolic device. In the sum of two waves (compressional and shear) emanating from the \(F^{l-1,lm}\) source, the integral is (Ben-Menahem & Singh 1981, p. 156; Hudson 1980, p. 38)

\[\int_{r/\alpha}^{+\infty} \tau F^{l-1,lm}(t - \tau) d\tau. \quad (B.9)\]

We obtain the STF versions of the displacement function, using formulas (A.6), (A.9), and (A.15). For example

\[0_p^{(l)}(\omega) = \frac{(-i\omega)^{l-1} \exp(i\omega r/\alpha)}{4\pi\epsilon(\lambda + 2\mu)\alpha^{l-1}(l-1)!} \cdot \left\{ n_j N_{A_1} \mathcal{F}_{A_1}(\omega) \right\}

+ \sum_{k=1}^{l-1} \left[ \left( \frac{l}{2l+1} \right) (A^{lk} - B^{lk}) N_{A_{l-1}} \mathcal{F}_{A_{l-1}}(\omega) + B^{lk} n_j N_{A_1} \mathcal{F}_{A_1}(\omega) \left( \frac{\alpha}{r\omega} \right)^k \right]

+ \left[ (2l+1)!! \left( \frac{\alpha}{r\omega} \right)^l \left( n_j N_{A_1} \mathcal{F}_{A_1} - \frac{l}{2l+1} \cdot N_{A_{l-1}} \mathcal{F}_{A_{l-1}}(\omega) \right) \cdot \left( 1 + \frac{\alpha}{r\omega} \right) \right], \quad (B.10)\]

where \(\mathcal{F}_{A_1}^{(l)}\) is defined by (A.4). For \(l=1\) (B.10) corresponds to eq. 6.22 by Hudson (1980); for \(l=2\) it corresponds to eq. 6 by Haskell (1963) or to eq. 4.149 by Ben-Menahem & Singh (1981). We obtain the STFT functions in time domain similarly to transition from
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(B.6) to (B.8).

\[ o_{p_j}^{(l)}(t) = \frac{1}{4\pi r(\lambda + 2\mu)\alpha^{l+1} l! (l + 1)!} \cdot \left\{ n_j N_A \partial_t^{l+1} \mathcal{F}_A (t - r/\alpha) \right. \]

+ \sum_{k=1}^{l-1} \left[ \left( \frac{l}{2l+1} (A_{lk} - B_{lk})N_{A_{l-1}} \partial_t^{l-k-1} \mathcal{F}_{f_{A_{l-1}}} (t - r/\alpha) \right) \left( \frac{\alpha}{r} \right)^k \right] \]

+ \left[ (2l+1)! (\frac{\alpha}{r})^{l+1} \left( n_j N_A \int_{r/\alpha}^{\infty} \tau \mathcal{F}_A (t - \tau) d\tau \right) - \frac{l}{2l+1} N_{A_{l-1}} \int_{r/\alpha}^{\infty} \tau \mathcal{F}_{f_{A_{l-1}}} (t - \tau) d\tau \right] \right\}, \quad (B.11a) \]

\[ o_{\delta_j}^{(l)}(t) = \frac{1}{4\pi \mu \beta^{l+1} l! (l + 1)!} \cdot \left\{ \left[ N_{A_{l-1}} \partial_t^{l+1} \mathcal{F}_{f_{A_{l-1}}} (t - r/\beta) - n_j N_A \partial_t^{l+1} \mathcal{F}_A (t - r/\beta) \right] \right. \]

+ \sum_{k=1}^{l-1} \left[ \left( \frac{l}{2l+1} \cdot A_{lk} + l \cdot B_{lk} \right) N_{A_{l-1}} \partial_t^{l-k-1} \mathcal{F}_{f_{A_{l-1}}} (t - r/\beta) \right]

\[- B_{lk} n_j N_A \partial_t^{l-k-1} \mathcal{F}_A (t - r/\beta) \left( \frac{\beta}{r} \right)^k \right] \]

\[- \left[ (2l+1)! (\frac{\beta}{r})^{l+1} \left( n_j N_A \int_{r/\beta}^{\infty} \tau \mathcal{F}_A (t - \tau) d\tau \right) - \frac{l}{2l+1} N_{A_{l-1}} \int_{r/\beta}^{\infty} \tau \mathcal{F}_{f_{A_{l-1}}} (t - \tau) d\tau \right] \right\} \], \quad (B.11b) \]

The sum of the expressions (B.11a) and (B.11b) for the values of \( l = 1 \) and \( l = 2 \) can be compared with those quoted by Aki & Richards (1980, eqs (4.27) and (4.30)) and by Kennett (1983, eqs (4.47) and (4.50)), respectively.

Similarly to (B.11a, b), for the spherical displacements of the class \( \eta = 2 \) it is easy to obtain

\[ \delta_j^{(l)}(t) = \frac{1}{4\pi (\lambda + 2\mu)\alpha^{l+1} l!(2l+3)} \cdot \left\{ - n_j N_A \partial_t^{l+1} \mathcal{D}_A \right. \]

+ \sum_{k=1}^{l-1} \left[ \left( \frac{l}{2l+1} (B_{lk} - A_{lk}) N_{A_{l-1}} \partial_t^{l-k+1} \mathcal{D}_{f_{A_{l-1}}} - B_{lk} n_j N_A \partial_t^{l-k+1} \mathcal{D}_A \right) \left( \frac{\alpha}{r} \right)^k \right] \]

\[- (2l+1)! (\frac{\alpha}{r})^{l+1} \left[ n_j N_A \left( \partial_t \mathcal{D}_A + \frac{\alpha}{r} \mathcal{D}_A \right) - \frac{l}{2l+1} N_{A_{l-1}} \left( \partial_t \mathcal{D}_{f_{A_{l-1}}} + \frac{\alpha}{r} \mathcal{D}_{f_{A_{l-1}}} \right) \right] \right\} \], \quad (B.11c) \]
where $\mathcal{D}_A = \mathcal{D}_A(t - r/\alpha)$ is given by (A.13), and

$$
2s_j^{(I)}(t) = \frac{l}{4\pi r \mu \beta^{l+1}(l+1)! (2l+3)} \cdot \left\{ \left[ \frac{N_{A_{l-1}} \partial_{t}^{l+1} \mathcal{D}_{jA_{l-1}}}{l_1 N_{A_{l-1}} \partial_{t}^{l+1} \mathcal{D}_{A_{l-1}}} - n^l_j \partial_{t} \mathcal{D}_{jA_{l-1}} \right] + \sum_{k=1}^{l-1} \left[ \left( \frac{(l+1) \cdot A^{lk} + l \cdot B^{lk}}{2l+1} \right) \cdot \mathcal{D}_{jA_{l-1}} - B^{lk} n^l_j \partial_{t} \mathcal{D}_{jA_{l-1}} \right] \cdot \left( \frac{\beta}{r} \right)^k \right\}
$$

$$
-(2l+1)! \left( \frac{\beta}{r} \right)^l \left[ n^l_j \partial_{t} \mathcal{D}_{jA_{l-1}} + \partial_{t} \mathcal{D}_{jA_{l-1}} \right] \cdot \left( \frac{\beta}{r} \right)^l \left( \frac{\beta}{r} \partial_{t} \mathcal{D}_{jA_{l-1}} \right) \right\},
$$

(B.11d)

where $\mathcal{D}_A = \mathcal{D}_A(t - r/\beta)$.

The STF toroidal displacement function is

$$
\text{1st toroidal displacement function is}
$$

$$
\text{2nd toroidal displacement function is}
$$

To test the validity of our transformations, we obtain the displacement function for the vector Helmholtz equation (Biedenharn & Louck 1981, eq. 7.6.10) by taking $\alpha = \beta$ and combining $p$ and $s$ contributions in (B.11a) and (B.11b). As another test of our derivations, we obtain the static displacement vectors from dynamic, temporal displacement functions by adding contributions (B.11a) and (B.11b) and by using the constant value of $\mathcal{F}_{A_{l-1}}$ in the above formulas and in (B.11e). All time derivatives of $\mathcal{F}_{A_{l-1}}$ in (B.11a), (B.11b) and (B.11e) disappear and we obtain the values of the static displacements $\mathcal{F}_{A_{l-1}}^{(I)}$ from eqs 2.5 and 2.7 of H1, respectively. A similar procedure with $\mathcal{F}_{A_{l-1}}^{(I)}$, applied, for example, to (B.11c) and (B.11d) transcribed in vector spherical harmonics, yields (see Eq. A.5)

$$
\text{2nd toroidal displacement function is}
$$

This is only part of the static solution (see eqs B.10a, and 2.2 in H1). The second part of the solution is obtained from the sum of the $k = 1$ terms in (B.1b) and (B.1c) (cf. Eqs 2.9). We obtain the full expression for static $\mathcal{F}_{A_{l-1}}^{(I)}$ (see eq. 2.8 in H1) by repeating transformations like that of (B.11a) and (B.11b) and adding the (B.12) result.