Spatial distribution of earthquakes: the three-point moment function

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Summary. The three-point moment of the spatial distribution of shallow earthquakes is determined for one local and one world-wide catalogue. We compare the numbers of hypocentre triplets forming a particular triangle in a real catalogue and in a randomized (simulated Poissonian) catalogue, which has the same boundaries and depth distribution of hypocentres as the real catalogue. The ratio of these quantities seems fairly well approximated by the reciprocal of the surface area of the triangle. There is good evidence that this ratio is independent of the form of the triangle once the surface area is fixed. These results, similar to those from the two-point spatial moments, indicate lack of any intrinsic scale of distance or triples configuration for distances between hypocentres ranging from a few kilometres up to 1000 km. The results, together with similar ones from two- and four-point moment studies, place limits on the possible models of earthquake fault geometries.

1 Introduction

This paper describes the continuation of our investigation into the stochastic geometry of earthquake faulting. In the first paper of this series (Kagan & Knopoff 1980a) we examined the two-point moment (or correlation function) of the spatial distribution of earthquakes. If $R$ is defined as the distance between two hypocentres (or epicentres), then our results showed that for shallow earthquakes this two-point moment could be approximated by a function proportional to $1/R$ for distances in the range of 0–2000 km. This hyperbolic law means that with regard to second-order moment properties, the spatial distribution of earthquakes lacks any intrinsic distance scale.

In order to continue our study of the spatial distribution of earthquakes, we now analyse the additional information from higher-order moments of this spatial pattern. Since the number of calculations increases approximately as $n^k/k!$, where $n$ is the number of events in a catalogue, and $k$ the moment's order, it is clear that this approach is feasible only for small values of $k$ and $n$. Earthquake hypocentres are distributed in a three-dimensional space;
therefore configurations containing up to four hypocentres are especially interesting for the study of the stochastic geometry of fractures. We investigate the three-point spatial moment in this paper and the four-point moment in the final report of this series. For the sake of brevity we refer to other papers of this series: Kagan & Knopoff (1980a) and Kagan (1981), as P2 and P4, respectively.

The large computation time mentioned above is not the only difficulty encountered in the study of higher-order moments. Other problems include a multidimensionality of the final results which makes it difficult to visualize them. An example of this multidimensionality is the fact that a three-point moment function depends on three distances between the points, in the case of an isotropic and homogeneous spatial distribution of earthquakes. As a consequence of these difficulties we do not yet have sufficient knowledge of the properties of higher-order moments. As a matter of fact we find very few references to this problem (the two volumes by Monin & Yaglom 1971, 1975, contain the most comprehensive information for stochastic continuous processes), and only one case of a detailed study of a three-point moment for a multidimensional stochastic point process (Peebles & Groth 1975; Groth & Peebles 1977; Groth et al. 1977). Hence we regard the present study as an exploratory attempt and consider it necessary to provide more technical details of computation and analysis than is common in the literature.

We want to comment briefly on our use of the terms 'correlation function' and 'moment function' (see also Appendix). The former designation belongs to continuous random processes, its equivalent for a stochastic point process being a second-order, cumulant measure (density) (Daley & Vere-Jones 1972; Kagan 1973). Cumulant measures are defined by moment measures of the same and lower orders. In the study of continuous processes we should distinguish between the order of a moment (cumulant) function, and its type or point number (Monin & Yaglom 1971). For the orderly — not related to the order defined above — point processes (Daley & Vere-Jones 1972) which are discussed in this paper, the order number is equivalent to the point number. Since the term 'correlation function' is widely known, we occasionally use this expression as well. Finally we will often use the shorter term moment instead of the more exact terms, moment density or moment measure.

2 Data description

In the study of higher-order moments, the problems with the original data (catalogues of earthquakes) involve not the quantity of information — because of the large amount of calculation required we are usually unable to employ all the data in catalogues — but rather the quality. In the catalogues, among the most important features that we take into account is the ratio of a linear dimension of the area covered by a catalogue to average errors in hypocentre (epicentre) location. This ratio should be higher for the three-point moment study than for that based on two-point moments (P2), since all three distances between the points should be larger than location errors and significantly smaller than the above linear dimension. Six catalogues were studied in P2; only two of them can be used for the present research: the Central California (USGS) local catalogue (McHugh & Lester 1978, and the references contained therein) and the world-wide NOAA catalogue (Meyers & von Hake 1976). In the case of the world-wide catalogue, the maximum distance accepted is determined by the distance corresponding to an abrupt change in the pattern of the spatial distribution of earthquakes. According to our results (P2) this change occurs at a distance of about 200 km and apparently corresponds to the average size of major tectonic plates.

The width of the area spanned by the USGS catalogue is about 100 km and the maximum distances, $R_{\text{max}}$, is 378 km (P2). The location errors are about 0.5 km for the USGS
Spatial distribution of earthquakes

catalogue (P2); for the NOAA catalogue these errors are more difficult to evaluate, but for our purposes the most significant issue is the vertical error in hypocentre location (P2). For instance, this error is so great for shallow events (in this study we use the depth limits of 0–35 km) that the depth information for these earthquakes is not used in this study. This amounts to the assumption of an error estimate of about 35 km.

The time period covered by the NOAA catalogue is 1965–77, the total number of events with \( m_b \geq 5.3 \) is 3888 (see also table 1 in P2). We have added data from 1976 to the USGS catalogue (McHugh & Lester 1978); the catalogue combined contains 7340 events for the years 1971–76 with lower magnitude cut-off \( M_L = 1.5 \) (cf. table 1 in P2).

3 Cumulative moments

In general, our study of the three-point moment for the spatial distribution of earthquakes follows the development proposed by Peebles & Groth (1975). For reasons explained in the Appendix we have not studied the cumulants of the process but rather the conditional moments.

We attempt to avoid the subjective judgement involved in arbitrary smoothing of our estimates of distribution density (Ripley 1977). To this end, as an estimate of the second and third-order properties of the process, we first tried to use the cumulative numbers of pairs of hypocentres, \( N_2(R) \), or the cumulative numbers of triplets of events, \( N_3(R) \), situated at a distance, or distances less than \( R \) from each other. Note that here and in P4 the subscript of \( N \) indicates the moment order, not the dimensionality of the phase space, as it did in P2.

The relation between the quantities \( N_k(R) \) and the conditional moment densities is seen from equation (A5). As shown in this formula, in order to normalize the computed numbers of pairs (or triplets, etc.) we should divide this number by an appropriate element of volume. The calculation of this volume element becomes a complicated problem of integral geometry in the case of three (cf. Santalo 1976, p.16) or four points. The further difficulties involve the edge effects due to boundaries of the area covered by a catalogue and inhomogeneities of depth distribution of earthquakes. The only feasible way to solve this problem in the case of higher order moments is to use the Monte Carlo simulation method. Thus we divide \( N_k(R) \) by \( N^p_k(R) \), the corresponding number of pairs or triplets in a simulated Poisson catalogue which has the same boundaries and depth distribution of hypocentres as the real catalogue:

\[
q_2(R) = \frac{N_2(R)n_p(n_p-1)}{[N^p_2(R)n(n-1)]}
\]

and

\[
q_3(R) = \frac{N_3(R)n_p(n_p-1)(n_p-2)}{[N^p_3(R)n(n-1)(n-2)]}
\]

where \( n_p \) is the total number of events in the synthetic catalogue. The normalization has an additional advantage of taking into account the total number of events in a catalogue. It means that the quantities \( q_k \) can be regarded as ratios of conditional and unconditional intensities (see equation A1):

\[
q_2(R) = \frac{m(y | x)}{[m(y)]}
\]

and

\[
q_3(R) = \{m_2(y, z | x) / [m(y)m(z)]\}^{(s)}
\]

Here, \{ \}^{(s)} means a symmetric sum:

\[
\{m_2(y, z | x)\}^{(s)} = m_2(y, z | x) + m_2(x, z | y) + m_2(x, y | z) + m_2(x, y | z)
\]

\[
\{m(y)m(z)\}^{(s)} = m(y)m(z) + m(x)m(z) + m(x)m(y)
\]
Thus this normalization facilitates a comparison of results from different catalogues.

It can be also shown that the two-point moment of the spatial distribution of epicentres is given by

\[ q_2(R) = nK(R)/S(R) \]

where \( K(R) \) is a second-order moment measure proposed by Ripley (1977), and \( S(R) \) is the total surface area of the circles of radius \( R \) centred on all epicentres in the catalogue. Only the part of any circle which is inside the area spanned by the catalogue is included in the sum. Therefore, equation (2) might be regarded as a special case of equation (1), which suits both hypocentral and epicentral spatial distributions.

For illustrative purposes the ratio (1) is multiplied by \((R/R_{\text{max}})^{k-1}\). For example,

\[ Q_2(R) = q_2(R) \cdot R/R_{\text{max}}, \]

\[ Q_3(R) = q_3(R) \cdot R^2/R_{\text{max}}^2. \]  

If the ratios \( q_k(R) \) are dependent on \( R^{1-k} \), \( Q_k \) will be constant and the plot of \( Q_k \) versus \( R \) will be a horizontal line. The ratios \( Q_2(R) \) and \( Q_3(R) \) are displayed in Figs 1 and 2, respectively. The two-point ratio for the NOAA catalogue was calculated previously (fig. 9a of P2). In Fig. 1 the first four curves represent the USGS subcatalogues selected with different values for the magnitude cut-off. These curves are all close to each other and indicate again (P2) that the spatial distribution of earthquakes is independent of magnitude. Minor differences between the ratios are probably caused by higher accuracy of hypocentre locations for stronger earthquakes (P2).

The middle portion of each of these curves is flat in the distance range of 0.5–50 km. These two limits roughly correspond: (1) to the magnitudes of location errors, and (2) to the dimensions of the area covered by the catalogue. The slight rise of the ratios at shorter distances (around 1 km) is probably caused by the limited time-span of the catalogue (6 yr). In order to demonstrate this we plot an average ratio for the USGS catalogue subdivided into 122 non-overlapping subcatalogues of 60 events each. The ratio \( Q_3(R) \) (the upper curve in Fig. 1) rises very strongly for small values of \( R \). We therefore conjecture that the ratio \( Q_3(R) \) will be close to a constant in a catalogue of long time duration. From equation (3) we see that if \( Q_3(R) \) is constant the ratio \( q_3(R) \) is proportional to \( 1/R \). This means that the conditional first-order moment has a distance dependence of \( 1/R \), exactly the dependence we would observe if all the hypocentres were distributed on an infinite plane (P2).

![Figure 1. Two-point cumulative hypocentral moments for the USGS catalogue and for plane simulations. Horizontal line in the plot corresponds to \( R^{-1} \) dependence of the moment.](image-url)
This last distribution of hypocentres may be illustrated by two synthetic catalogues in which hypocentres are constrained to locations on vertical and horizontal planes. The depth distribution of hypocentres on the vertical plane is simulated so as to correspond to a real distribution; the horizontal plane is not a realistic example, but it will be useful in our later deliberations (Section 6). The ratio $Q_2(R)$, for the simulation on the vertical plane, is independent of distance as soon as border effects disappear. In the case of a horizontal plane the non-uniformity of the depth distribution of hypocentres, in the Poissonian catalogue used for comparison (see equation 1), may also perhaps be an important factor influencing the behaviour of the lower curve in Fig. 1. This depth distribution has a length scale of kilometres (see fig. 5 in P2).

The length of the vertical plane on which hypocentres were simulated is about 300 km; if we decrease the length of the plane segment, the corresponding curve in Fig. 1 will move upwards. If we compare the vertical position of this curve with those corresponding to the real catalogue, we see that statistically the real earthquake distribution in the 1971–76 USGS catalogue is equivalent to a vertical plane of about 200 km in length. The comparison of results for the catalogue of small time-spans (upper curve in Fig. 1) with appropriate
results for the full catalogue demonstrates that as the time-span of a catalogue increases our ratio in the plot moves downwards. Thus, we may guess that for a sufficiently long time-span the catalogue would be equivalent to a vertical plane with a length equal to the maximum distance $R_{\text{max}}$ (cf. figs 1 and 2 in P2).

The picture is similar for the USGS three-point cumulative moment (Fig. 2a). The third curve — that for $n = 299$ — is obtained by randomly sampling the catalogue, with $M > 1.5$. All curves exhibit the same behaviour as in a two-point case; it is interesting to note that the value of $Q_3(R)$, for all but the curve with $n = 122 \times 60$, is approximately equal to the square of the corresponding value of $Q_2(R)$. This behaviour is to be expected, if, for instance, all points of a process are distributed on a plane with a uniform depth distribution. Then,

$$Q_k(R_{\text{max}}) = C(R_{\text{max}}) = 1;$$

as the distance approaches zero,

$$Q_k(0) = [C(0)]^{k-1}.$$  

For the USGS catalogue the quantity $C(0)$ is approximately equal to 0.2 (Fig. 1). The fact that the case for $n = 122 \times 60$ diverges from this pattern shows that on a small time-scale the spatial distribution of earthquakes is not at all equivalent to that of a plane.

Three-point cumulative functions for the NOAA catalogue (Fig. 2b) are similar to those of the USGS catalogue, the decline of the ratio at small distances is probably caused by location errors and the effects of projection of hypocentres on a horizontal plane (P2). These results show that in regard to the properties of the cumulative third-order moment, the self-similarity of the spatial distribution of earthquakes holds from a distance of a few kilometres up to 1000 km. This is similar to the second-order results (P2). Next we consider the three-point moment in greater detail. For this purpose we need to study a distribution that is a function of all the three distances separating the points, not only the maximum distance that we have analysed above.

### 4 Measurement of the three-point distribution density

As illustrated in Fig. 3, the three sides of a triangle formed by hypocentres are labelled $R_1$, $R_2$, $R_3$, and arranged in such a manner that $R_3 > R_2 > R_1$. For convenience, we introduce two new variables which are slight modifications of those used in a study by Peebles & Groth (1975):

$$U = R_1/R_2 \quad \text{and} \quad V = (R_1 + R_2 - R_3)/R_1.$$  

Both of these quantities have magnitudes in the range from 0 to 1.

The condition $R_3 > R_2 > R_1$ defines a segment of a circle or a sphere which we subdivide into $20 \times 37$ cells (Fig. 3a) in such a way that $U$ and $V$ change by uniform logarithmic intervals equal to $2^{1/6}$ and $2^{1/4}$, respectively. The maximum distance in the triangle, $R_3$, is subdivided into 19 or 37 intervals, each $2^{1/6}$ times larger than the previous one. The number of triplets of earthquakes which fall in each of these $(U, V)$ boxes is compared with the corresponding number of triplets in the Poisson catalogue; the normalized ratio is then displayed in the form of a perspective diagram (Fig. 3b) and a contour map (Fig. 3c). The
small triangles near the corners of these displays show schematically the form of the triangle corresponding to each corner.

The distribution \( V \times U \times R \) represents a four-dimensional plot which cannot be reproduced here in its entirety. Thus we show the ‘slices’ of four-dimensional histograms corresponding to several intervals of the maximum distance \( R_3 \). To avoid bias in the sampling and display of results, we use the same intervals of \( R_3 \) throughout this paper and P4.

We would like to use cumulative distributions (as was done in the previous section) to compare the numbers of triplets in the real and Poisson catalogues:

\[
q_3(V, U, R_3) = \frac{N_3(V > v, U > u, R > r) n_p(n_p - 1)(n_p - 2)}{N^P_3(V > v, U > u, R > r) n(n - 1)(n - 2)}
\]

The advantage of these ratios is that the resulting functions need not be smoothed, either by the choice of the interval size or by any other method (cf. Ripley 1977). Unfortunately, after obtaining these cumulative functions we discovered that they are difficult to interpret. Most probably, this happens because the triplets corresponding to small distances are included in the total sum \( N_3 \); these triplets then ‘contaminate’ the values of \( q_3 \) for larger values of arguments. As will be seen in Sections 5 and 6, the values of \( N_3 \) for small arguments are strongly influenced by location errors and other types of distortions. Thus, we replace (5) with the histogram of the ratio

\[
q_3(V_i, U_j, R_3) = \frac{N_3(v_{i+1} > V > v_i, u_{j+1} > U > u_j, r_{l+1} > R > r_l) n_p(n_p - 1)(n_p - 2)}{N^P_3(v_{i+1} > V > v_i, u_{j+1} > U > u_j, r_{l+1} > R r_l) n(n - 1)(n - 2)}
\]

Figure 3. Measurement of the three-point distance distribution and examples of displays. (a) Arrangement of the triangle sides (right plot) and subdivision of the measurement area into \( 20 \times 37 \) cells. (b) Perspective diagram of the smoothed histogram. (c) Isoline map of the smoothed histogram.
where $V_i$ and $U_j$ correspond to the mid-point of the $(V, U, R_3)$ interval. For illustrative purposes, we multiply the $N_3/N_2$ ratio by the function $f(U, V, R)$ which, in our assumption, is proportional to the reciprocal of the distribution density function $q_3(U, V, R)^{-1}$. As a consequence, if our assumption is correct, the final result should equal to a constant over the entire span of each of the arguments:

$$Q_3(V_i, U_j, R_3) = q_3(V_i, U_j, R_3) \cdot f(V_i, U_j, R_3)/f(1, 1, R_{\text{max}}).$$

The function of the denominator serves to normalize the ratio $Q_3$ and to make it dimensionless.

5 Results for the USGS catalogue

Table 1 illustrates the method described above. The table is a tabular representation of Fig. 3, with data from $2 \times 3$ intervals in the figure combined into one entry of the table in order to limit the table to one page. The first row of the table corresponds to the first $U$ interval in Fig. 3, the second row to the intervals 2–4 in the figure, etc. For $V$ the agreement is that one column in the table corresponds to two intervals of the figure: the first column of Table 1 is obtained from the combination of intervals 1 and 2 of Fig. 3(a), the second column Table 1.

<table>
<thead>
<tr>
<th>$U$ (km)</th>
<th>$V$ (km)</th>
<th>$W$ (km)</th>
<th>$Q_3(V, U, R_3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
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<td>3</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
</tbody>
</table>

Table 1. Calculation of normalized ratio $Q_3(V, U, R_3)$ in three dimensions.

POISSONIAN CATALOG ($n_p = 2\times100$)

<table>
<thead>
<tr>
<th>$h(km)$</th>
<th>0.00</th>
<th>0.10</th>
<th>0.20</th>
<th>0.30</th>
<th>0.40</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.21</td>
<td>1.72</td>
<td>2.44</td>
<td>3.47</td>
<td>4.99</td>
<td>7.28</td>
</tr>
</tbody>
</table>

$-\text{means that the value of the ratio is undefined due to the insufficient number of hypocentre triplets in Poissonian or real catalogues.}$
corresponds to the intervals 3–4 in that figure, etc. The correspondence between corners of the table and the shape of triangles is the same as shown in Fig. 3(c). The entries in the two upper parts of the table are the numbers of triplets in the USGS, and in the Poisson catalogue used for comparison. We add together two different simulations of the Poisson catalogue, with 1200 events in each.

The amount of calculation necessary is illustrated by the USGS subcatalogue which we use in Table 1 (the case with $M > 2.5, n = 803$). For this catalogue, $12.6 \times 10^6$ triplets were analysed with a total time expenditure of about 1000s on an IBM 370/3033 computer. The distance separating the points does not exceed 36.2 km in this computation.

The values of the triangle height, $h$, which correspond to each mid-point of the combined boxes (see above), are shown on the boundaries of the table near the boxes to which they correspond. The ratio $Q_3$ shown in the bottom third of the table is obtained by assigning

\[
\text{Figure 4. Normalized histograms for the USGS catalogue, } M > 2.5, n = 803. \text{ The arrows on the left of the figure, and thick lines on the right, indicate approximate positions of a triangle height of 0.5 km, or } h > 0.5 \text{ km, respectively. The numbers in the maps on the right correspond to values of the normalized ratio multiplied by 100. If the probability of a configuration of three hypocentres were inversely proportional to the area of the triangle defined by the three hypocentres, the normalized histograms would define a 'plateau' of constant values. (a) } 36.2 > R, > 32.3 \text{ km. (b) } 12.8 > R, > 11.4 \text{ km. (c) } 5.08 > R, > 4.53 \text{ km. (d) } 3.20 > R, > 2.85 \text{ km.} \]
In equation (6) the value \( S \), where \( S \) is the surface area of the triangle. As we explain in Section 8, this dependence will be obtained if all hypocentres are confined to one plane in three-dimensional space. This model is the most commonly used geometrical representation of an earthquake fault. The ratio \( Q_3 \) is then multiplied by \( 10^5 \) and rounded off to the nearest integer.

In parts of the table where \( h > 0.5 \) km, the values of \( Q_3 \) are more or less stable. Fig. 4 shows the ratio for several choices of \( R_3 \). The perspective diagrams on the left side of this and following figures are normalized in such a manner as to have the same maximum height. This means that only the shape of the smoothed histograms can be examined in these diagrams; the absolute values of the ratio \( Q_3 \) are displayed in the right parts of the plots. We see again that inside the area where \( h > 0.5 \) km, the values of \( Q_3 \) are more or less constant, rising slightly when \( R_3 \) decreases -- the same behaviour as with cumulative moments (Fig. 2).

In the next section we discuss several sources of non-uniformity of \( Q_3 \) values. Here we note that it is a general rule that statistical variations rise significantly with the increase of the moment order. Thus we should expect a higher level of the scatter than was the case for the two-point moment study (P2). Taking this into account, the independence of the normalized ratio from the values taken by all variables \( (U, V, R_3 \) or \( R_1, R_2, R_3) \) in Fig. 4, implies that the probability of any three hypocentres forming a particular triangular configuration depends only on the surface area of the triangle. This probability is proportional to \( 1/S \).

Peebles & Groth (1975, see also Groth & Peebles 1977) suggest another form for the three-point function. In our case, this form would correspond to

\[
\begin{align*}
\hat{q}_3(R_1, R_2, R_3) &= \frac{m_2(y, z|x)}{\left[m(y) m(z)\right]} \\
&= \frac{m(y|x) m(z|x)}{\left[m(y) m(z)\right]} \\
&+ \frac{m(x|y) m(z|y)}{\left[m(x) m(z)\right]} \\
&+ \frac{m(x|z) m(y|z)}{\left[m(x) m(y)\right]} \\
&\propto \left[R_1 R_2^{-1} + R_2 R_3^{-1} + R_1 R_3^{-1}\right].
\end{align*}
\]

In this formula, \( x, y, z \) are coordinates of hypocentre positions, and \( R_1, R_2, R_3 \) are distances between these three points. This formula implies that the ratio \( q_3 \) is proportional to the reciprocal of the surface area of a rectangle. As its sides, this rectangle has the radii of inscribed and circumscribed circles of the triangle formed by three hypocentres.

Table 2. Test of equation (7) approximation of three-point moment.

<table>
<thead>
<tr>
<th>( h ) (km)</th>
<th>( v )</th>
<th>( l )</th>
<th>( w )</th>
<th>( y )</th>
<th>( u )</th>
<th>( w )</th>
<th>( y )</th>
<th>( u )</th>
<th>( w )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.03</td>
<td>0.03</td>
<td>0.03</td>
<td>0.05</td>
<td>0.60</td>
<td>0.86</td>
<td>1.21</td>
<td>1.72</td>
<td>2.44</td>
<td>3.87</td>
</tr>
</tbody>
</table>

In this table, the values of the ratio along
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the U-axis are approximately the same as in the previous case, whereas along the V-axis the values are definitely not constant, demonstrating that approximation (7) is invalid for our data. The implications of this result will be considered in Section 8.

6 Dependence of the moment on location error, time and magnitude

In this section we try to show that deviations from the $1/S$ law are caused by different external factors. First we check the location errors which have already been partially discussed. Two catalogues have been generated by the Monte Carlo procedure. In these catalogues the hypocentres are constrained to a vertical plane; the depth distribution is simulated in such a way as to correspond to that of the real, USGS catalogue. The points of one set are then displaced by a random vector perpendicular to the plane. The length of this vector is taken to be a normally distributed variable with a standard deviation $\sigma$ equal to 0.5 km. Three-point distributions like those shown in Table 1 have been computed for both catalogues and normalized in a standard manner as in equation (6). The difference between the ratios $Q_3$ from these two catalogues is displayed in Fig. 5.

We see that the random shift of the points causes the resulting distribution to drop significantly for small values of $U$ and $V$, with a slight increase of the moment densities toward medium-range values of the arguments. This effect becomes more pronounced - but more difficult to simulate because of computation costs - when the maximum distance, $R_3$, decreases.

In Fig. 4 we see a similar decline of the ratio at small values of $V$ and $U$ for real earthquakes. The noticeable exceptions to this rule are the values of the ratio in the corner of a diagram with small $U$ and high $V$ values. As we see from Fig. 3(b and c) this corner corresponds to an isosceles triangle with a small third side. The ratio in such locations exhibits a pronounced rise which can apparently be explained in a manner similar to that in Section 3: by the insufficient time-span of the catalogue. To clarify this point we calculate the ratio values for the USGS catalogue subdivided into 122 subcatalogues. The plots in Fig. 6 exhibit a very strong rise in the upper right corner, especially for large $R_3$. For small distances the influence of location errors apparently predominates, thus the diagram (Fig. 6c) is similar to that of the full catalogue (Fig. 4c) except for overall, higher values of the ratio $Q_3$.

We can see these features in more details in Table 3. In the top part of the table the high values of the ratio are also concentrated in the upper right corner. The most reasonable

![Figure 5](image)

Figure 5. Influence of location errors ($\sigma = 0.5$ km) for hypocentres located on a vertical plane; $36.2 > R_3 > 32.3$ km. Negative values signify that the number of triplets decreases due to location errors.
Figure 6. Normalized histograms for the USGS catalogue subdivided in 122 non-overlapping subcatalogues ($M > 1.5, n = 122 \times 60$). (a) $36.2 > R > 32.3$ km. (b) $12.8 > R > 11.4$ km. (c) $5.08 > R > 4.53$ km.

Table 3. Time dependence of normalized ratio $Q_i(V, U, R_i)$.  

<table>
<thead>
<tr>
<th>$R$ (km)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<th>8</th>
<th>9</th>
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<tbody>
<tr>
<td>$Q_i$</td>
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<td>USGS CATALOG ($M \geq 1.5, n = 132 \times 10^6$)</td>
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</table>
Spatial distribution of earthquakes

Explanation of this pattern is that the concentration of seismicity is restricted to small areas ('hot spots') of a fault. When the time-span of a catalogue increases these spots expand and cover the whole area of the catalogue. To demonstrate it, in Table 3 we also show the ratios which are computed for the USGS catalogue with aftershocks, at least partially, deleted from the catalogue. This removal is effected in the following way (cf. Gardner & Knopoff 1974). A time-distance window is defined for times following, and locations surrounding each earthquake; the time limit and radius of the sphere are calculated from

\[ T(M) = 935 \times 3^{M-7.0} \text{ (day)} \]

and

\[ R(M) = 300 \times 3^{M-7.0} \text{ (km)} \]

(Kagan & Knopoff 1980b), where \( M \) is the magnitude of an earthquake \( - M_L \) in this case. If the window includes another earthquake, then this new earthquake is included in the 'aftershock sequence' and we start to look for other events in both windows. The process continues until no more additional earthquakes can be found in all the windows corresponding to the shocks included in the sequence. At this point we remove all earthquakes but the first one from this interlinked cluster, but we assign this event the magnitude value of the strongest shock in the sequence.

In creating this new catalogue, with aftershocks deleted, we hope to reduce the influence of the time-span of the catalogue; Table 3 shows that, at least partially, this has been accomplished. In comparison to the entries in Table 1 the ratio in the lower part of Table 3 drops significantly in the upper right corner. We would also expect the values in the lower left corner to become closer to that of the corner of the diagram corresponding to an equilateral triangle (see Fig. 3). Apparently it does not happen. However the effect is small — the normalized ratio drops only by about a factor of two while \( h \) changes by a factor of 30. This might be due to statistical variations, to inadequacy of our \( 1/S \) model, or finally to errors not yet taken into consideration.

One of the likely sources of error is the possible anisotropy of earthquake faults, i.e. preferred orientation of the faults in the area under study. To test this idea, we use the approach described in Section 3, namely two catalogues have been generated in which earthquake foci are constrained to lie in either a vertical or a horizontal plane. Three-point moments obtained for these catalogues are displayed in Fig. 7. The raised edges of the diagrams for \( U \) and \( V \) close to zero are caused by our normalization procedure, in which the values \( q_3 \) are multiplied by the estimate of the triangular area \( S \) for the mid-point of the intervals shown in Fig. 3(a). Clearly, the centre of the distribution does not always correspond to the centre of the cell, and this discrepancy should be strongly in evidence for intervals corresponding to small values of \( U \) and \( V \). Due to the small number of triplets in these intervals, \( 4 \times 6 \) combined cells have to be used, which exacerbates the problem.

The most conspicuous feature of the plots shown in Fig. 7 is the lack of data corresponding to an equilateral triangle for the simulation with hypocentres on a vertical plane. This is easily explained by the fact that in the USGS catalogue hypocentres are located above a depth of 20 km. Therefore, an equilateral triangle with a height of more than 20 km or a side length of more than 23 km is not possible on the vertical plane. For the simulation with foci on a horizontal plane, the picture is just the reverse — a relative excess of the number of equilateral triangles. It is possible that other modifications of the distribution histogram occur, but they are hard to discern because of statistical variations due to the small size of the synthetic catalogues. This size is limited by the expense of counting hypocentre triplets in computations with simulated catalogue.
The last examination we carry out with the USGS catalogue concerns the dependence of the three-point moment function on magnitude. In Fig. 8 the ratio $Q_3$ is plotted for two subsets of this catalogue: in the first, 299 earthquakes are sampled randomly from the full catalogue with $M > 1.5$; the second subcatalogue includes all earthquakes with $M > 3.0$. When we compare these plots with the corresponding slice for the subcatalogue with $M > 2.5$, which is shown in Fig. 4(a), we see that all three distributions have approximately the same form, with the diagrams corresponding to stronger earthquakes having higher values of the ratio in the upper right corner of the plot. As explained in P2, these high $Q_3$ values are most likely caused by a higher location accuracy for stronger shocks coupled with a tendency for the shocks to occur in 'hot spots'. This result serves as an indication, additional to that in P2, that the spatial distribution of earthquakes is independent of magnitude.

Figure 7. Dependence of three-point moment on orientation of planes used for catalogue simulations; $36.2 > R_z > 32.3$ km. (a) Vertical plane. (b) Horizontal plane.

Figure 8. Dependence of three-point moment on the magnitude limits in the USGS catalogue; $36.2 > R_z > 32.3$ km. (a) $M > 1.5$, $n = 299$. (b) $M > 3.0$, $n = 254$. 
7 Results for the NOAA catalogue

We continue our analysis of the three-point moment function by testing whether this pattern of $1/S^3$ dependence continues with an increase of distance between foci. In Table 4 we display the numbers $N_3$ and the ratio $Q_3$ for the world-wide NOAA catalogue of shallow earthquake epicentres. The comparison of the upper parts of Tables 1 and 4 shows that the projection of hypocentres in NOAA catalogue on a horizontal plane significantly changes our distribution of $N_3$ values.

The use of epicentres -- not hypocentres -- and the absence of boundaries in the NOAA catalogue make it possible to calculate the expected number of triplets directly, without simulation of the Poisson catalogue. The outline of this computation is as follows. We ignore effects of the non-sphericity of the Earth and consider the comparison catalogue as a set of random points on a surface of the spherical earth. On this surface, the cumulative number of triplets with a maximum distance between points equal to $R_3$, is given by

$$N_3^p(R_3) = n_p(n_p - 1)(n_p - 2) \left(\frac{s(R_3)}{s_0}\right)^2 \cdot P.$$  

In this formula the product $n_p(n_p - 1)(n_p - 2)$ represents the total number of triplets in a catalogue with $n_p$ events, $s(R)$ is the surface area of a spherical cap; this cap has a radius $R$ on the surface of the sphere. The quantity $s_0$, in above equation, is the total surface area of the spherical earth. By $P$ we designate a probability that the distance between two points is smaller than $R$. These points are to be chosen randomly in a circle, or on a spherical cap of radius $R$, provided that this radius is much smaller than the radius of the sphere. This probability may be calculated by a formula analogous to (4) in P2. The computation yields the value of 0.587 for $P$.

Calculation of the expected number of triplets in any of the cells of Fig. 3(a) is effected by means of

$$N_3^p(R_{3l}) = \left[ N_3^p(r_{j1}) - N_3^p(r_{j}) \right] S_c/S_b.$$  

Table 4. Calculation of normalized ratio $Q_3(V, U, R_s)$ in two dimensions.

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<td>10788</td>
<td>16572</td>
<td>39099</td>
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**NORMALIZED RATIO ($\times 10^3$)**

| $h(kn)$ | 17.1 | 24.2 | 34.3 | 46.6 | 68.9 | 98.2 | 141.1 | 206.0 | 312.4 | 521.1 |

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Here, $S_a$ is the surface area of the whole area of measurement and $S_c$ is the surface area of a cell, which is computed by numerical integration.

Four slices of the three-point ratio $Q_3$, calculated for the NOAA catalogue, are shown in Fig. 9. As we explained in Section 2, for the NOAA catalogue we expect major sources of distortion to arise from our use of epicentres, rather than hypocentres. The arrows and thick lines in Fig. 9 indicate the position of $h = 35$ km, or $h > 35$ km, respectively. This 35 km limit is explained in Section 2.

It is evident that the ratio $Q_3$ is more or less constant again, but in this case over the range of distances from 80 to 1000 km. The decline of the ratio for small $V$ is most probably connected with errors due to projection of hypocentres on to a horizontal plane. Unfortunately, here it is too difficult to estimate the effects of the projection on the ratio values, although this was possible for the case of two-point moments treated in P2.

8 Discussion
The results in the previous sections indicate that the density of the three-point spatial moment of earthquake hypocentres can be fairly well approximated by a simple formula, in
which the density is inversely proportional to the triangular area formed by each set of three hypocentres. The results seem to rule out approximation (7) which actually decomposes the conditional second-order moment in terms of a symmetrical sum of products of conditional first-order moments. As we mentioned above, this decomposition seems to be appropriate for the three-point moment of the spatial distribution of galaxies (Peebles & Groth 1975; Groth & Peebles 1977). A similar decomposition also exists for four-point cumulants of the galaxy spatial pattern (Fry & Peebles 1978). To some degree the situation is analogous to factorization of unconditional moments of the Poissonian process into the products of first-order moments (see Appendix). Mandelbrot (1975, 1977) proposes an isotropic random walk as a model of the galaxy distribution and shows that the three-point moment function for this model should equal to a symmetrical sum of two-point moments. Thus this decomposition may indicate that all statistical connections of points in a set are produced by a binary interaction of events, an isotropic random walk is one of the examples of such interaction.

Here, we discuss model (7) in detail because we have indications that the binary interaction, in the form of a critical branching process, is a good description of time-magnitude properties of the earthquake occurrence (Kagan & Knopoff 1981). However, the problem is that an earthquake focus should be represented geometrically in tensor form as an elementary dislocation. Hence in reality the second-order moment, or the conditional first-order moment, should be a tensor of the fourth rank. In general, kth-order moments correspond to the tensors of rank 2k (Shermergor 1971; Kroener & Koch 1976).

It is, of course, possible that these higher-order tensor moments could be factorized in the manner described above, i.e. in terms of conditional first-order tensor moments. However, the information contained in the available catalogues describes the earthquake focus as a scalar point characterized only by location coordinates. This means that, with these catalogues, only scalar versions of the spatial moments can be studied at present. Thus we come to the conclusion that even if it is sufficient to know only the pairwise interaction of earthquake fault planes, scalar moments of higher order than those studied in the papers of this series would be required to describe the stochastic interrelations of fault planes properly. We summarize the above discussion by stating that the rejection of the approximation of the three-point moment in equation (7) for the scalar spatial distribution of earthquakes does not preclude the possibility that a similar decomposition could exist for a tensor form of the distribution. It means that there is no contradiction between the results of this paper and our previous research (Kagan & Knopoff 1981).

As we stated in P2, the results for the two-point moment rule out the model of spatial pattern of earthquakes based on an isotropic, homogeneous distribution of hypocentres in three-dimensional space (see also the Appendix). Now, using the results for the scalar, third-order moment obtained in this paper, we can further restrict the possible choice of geometrical models for earthquake faults. For example, an isotropic random walk, in which each step has a Gaussian variable as its projection on any axis, has a fractal dimension of two (Mandelbrot 1977). Thus this model would be consistent with our two-point moment results (for additional details see P2). As was explained above, this model should be rejected now on the basis of the three-point results. However, this conclusion is of little practical value, because it is clear even from a visual observation that the random walk is not an appropriate model for the earthquake spatial data.

The most commonly used model of an earthquake fault, that of hypocentres occurring on an isolated plane, is still consistent with 1/S behaviour of the three-point moment. We can see this from Monte Carlo simulations of the distribution of hypocentres on a vertical plane (Fig. 7a) and also from the simple arguments which follow. If all points of a set are
constrained to lie on an infinite plane, then the probability of finding a focus at a distance $R$ from a given point is proportional to $2\pi R$. If points are distributed randomly in three dimensions this probability is then proportional to $4\pi R^2$. If the third point is at a perpendicular distance $h$ from the line connecting the first two points, the corresponding probabilities for two and three dimensions would differ by a factor of $h$. When we normalize the three-point distribution we divide it by three-dimensional values (cf. equation 1), and therefore obtain the $1/S$ dependence. This dependence means that given two hypocentres, the conditional intensity, or the conditional number of points per unit of volume, is proportional to $1/h$.

There are additional consequences of the NOAA results. Because we are considering earthquakes in the narrow depth limits of 0–35 km, their distribution may be well approximated by lines or thin belts in two dimensions; the $1/S$ dependence means that given two epicentres, the conditional number of epicentres increases as $1/h$ as we approach the line connecting these points. This result clearly contradicts the model of earthquake faulting which consists of one isolated line, since this model results in the dependence of the moment in the form of a Dirac delta function $\delta(h)$. Even if we consider a distribution of epicentres not on one isolated line, but ‘diffused’ along seismic belts, the three-point distribution of distances should exhibit high values of the ratio for $V$ close to zero. The corresponding diagrams in Fig. 9 demonstrate just the opposite behaviour. A similar problem arises in the analysis of the four-point moment function, thus detailed discussion will be more appropriate in P4.

Acknowledgments

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References


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Appendix

Here we outline some basic definitions of the moment structure of stochastic point processes and discuss methods of the statistical estimation of moment measures used in papers of this series (P2, this work and P4). See Daley & Vere-Jones (1972), Kagan (1973), Ripley (1977), and Vere-Jones (1978) for more detail.

The first-order moment measure of a stochastic point process can most easily be perceived as a probability, Pr, of finding an event in a very small volume dx surrounding the point x:

\[ M_1(dx) = Pr[N(dx) = 1], \]

where \( N(A) \) is the number of events in volume \( A \). Later we will usually omit subscript 1 for the first-order moment. Here we consider only those cases in which the measure is always continuous and hence has a density

\[ M(dx) = m(x) dx. \]

For a homogeneous stochastic process the density \( m(x) \) is a constant called the intensity of a process \( m = \lambda \).

Similarly, the second-order moment measure is defined as

\[ M_2(dx, dy) = Pr[N(dx) = 1, N(dy) = 1] = m_2(x, y) dx dy. \]

This moment can be determined through a conditional first-order moment or conditional intensity of a process (Cox & Lewis 1966):

\[ m_2(x, y) = m(y | x) m(x). \] (A1)

In this formula \( m(y | x) dy \) means the average number of objects in the interval \( dy \), if a point \( x \) is occupied by another event. The ratio \( m(y | x) m(y) \) has a very simple intuitive meaning:
it shows by what factor the existence of the event at a point $x$ increases the probability of another event at a point $y$. In most physical systems a statistical interaction between events diminishes and then disappears as the distance $r$, increases:

$$m(y|x) \rightarrow m(y) \quad \text{as} \quad r \rightarrow \infty. \quad (A2)$$

Cumulant measures of a process can be defined as follows

$$C_i(dx)/dx = c(x) = m(x)$$

$$c_2(x,y) = m_2(x,y) - m(x)m(y)$$

$$c_3(x,y,z) = m_3(x,y,z) - m_2(x,y)m(z) - m_2(x,z)m(y) - m_2(y,z)m(x) + 2 \cdot m(x)m(y)m(z).$$

The quantity analogous to the second-order cumulant density, $c_2(x,y)$, is called a covariance, or correlation function in the theory of continuous random processes (Monin & Yaglom 1971, 1975). The advantage of the use of cumulant measures, or densities, is that if a point process consists of groups or clusters of events, each group having one, two, etc., members, then the cumulant measures show unambiguously the group content of the process (Kagan 1973). For instance, for the Poissonian, homogeneous process which consists only of ordinary points, all cumulants but the first one are equal to zero, whereas the moments are given by

$$m_k = \lambda^k.$$

If a homogeneous process consists of independent groups (or clusters) of up to order $l$, cumulants of $(l+1)$th and higher orders would be equal to zero. This means that the moments of $(l+1)$th and higher orders could be expressed in terms of moments of the order $l$ and lower. The random process is then defined completely if cumulants or moments up to the $l$th order are known. This observation explains why cumulant functions are often used in the analysis of point processes (cf. Peebles & Groth 1975; Groth & Peebles 1977).

The two-point correlation function $w$, which is studied in the above-mentioned works, could be expressed as follows in the case of an homogeneous spatial random process

$$w(r) = \hat{c}_2(x,y)/[\hat{c}(x)\hat{c}(y)] = \hat{m}(y|x)/\hat{m}(y) - 1, \quad (A3)$$

where $r$ is the distance between points $x$ and $y$, and the carats denote statistical estimates of corresponding quantities.

If we try to apply the discussed standard technique to the statistical analysis of the spatial distribution of earthquakes, we encounter an immediate problem because of unusual properties of this distribution. As we have shown in P2, the density of the conditional first-order moment $A1$, has the dependence

$$m(y|x) \propto r^{-1}$$

on the distance $r$ between hypocentres of shallow earthquakes (cf. equation A2). This hyperbolic law means that the spatial pattern of earthquake distributions belongs to a class of ‘self-similar’ random fields (Mandelbrot 1977), for which the unconditional moment density $m(x)$ is not well defined. The simplest model for such a process would be a set of points confined to an infinite plane in three-dimensional space. This model, as well as other models applied to explain the hyperbolic behaviour of the conditional intensity function (cf. Mandelbrot 1977) is not homogeneous. The implication of this result is that the standard assumption of homogeneity is most probably not tenable for earthquake spatial
distribution. We can clarify this by examining the statistical estimate of the unconditional intensity of the homogeneous process

$$\hat{\lambda} = N(W)/|W|,$$

(A4)

where $|W|$ is the area of the region spanned by a catalogue. This area is not well defined because we cannot draw boundaries of the region unambiguously. If, for example, we move the south-west border of the USGS region (see figs 1 and 2 of P2) still further to the south-west, very few new earthquake hypocentres would be added, while the total area might expand significantly. This means that our estimate of $\lambda$ in (A4) can be made as small as possible; thus, in a cumulant estimate, (A3), the first term is defined only up to a multiplicative factor, whereas the second one is a constant. On the other hand, the estimate of the conditional moment is independent of the area $W$:

$$m(r, r + \Delta r) = \frac{N_2(r + \Delta r) - N_2(r)}{S(r, r + \Delta r)(n - 1)},$$

(A5)

where $N_2(r)$ is the total number of event pairs with a distance range smaller than $r$, and $S(r_1, r_2)$ is the volume of a spherical shell with internal and external radii $r_1$ and $r_2$. This observation is in agreement with Mandelbrot's (1977) suggestion that for self-similar random processes only conditional probabilities are well defined and should be the subject of studies. Similar arguments can be made about the three-point cumulant and moment functions of the earthquake spatial distribution.

The discussion above explains why our series of papers does not have the one dedicated to the study of the first-order moment.